

The Geometrical Behavior of a Modified
Riemann Zeta Function without Analytic Continuation:
An Example of a Quasi-Convergent Series

Robert Duncan

The Riemann zeta function can be easily modified to do away with the need of the complex plane and analytic continuation, and yet, still find the zeros to the original function with high precision and with a finite number of terms. Other's have noticed some of this but only in passing and without in-depth exploration and understanding. This paper explores a modified zeta function leading to some fascinating discoveries.

The Generative Mathematics of the Modified Zeta Function:

The jumping off place is, of course the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

where 's' is a complex number, 'a + bi'. While it is not necessary to understand the steps taken to arrive at the modified zeta function, they are presented here, briefly. First, take the conjugate of 's' (a - bi), which puts the imaginary part of the summation (n^{bi}) in the numerator. Next, use Euler's formula to convert 'n^{bi}' to the trigonometric form cos(b*ln(n)) - i*sin(b*ln(n)). The denominator is still n^a. Now each term of the summation is a single complex number that can be viewed as an address in the complex plane. As long as there are no operations done on the addresses other than addition we can also view each term as a Cartesian address in the plane by dropping both the i and the negative sign: the x coordinate is the cosine part and the y coordinate is the sine part. The denominator of each coordinate is just n^a. And finally, applying the Pythagorean theorem (√(sin²+cos²)=1) we have a summation function whose terms are Cartesian addresses, which we can view as vectors in the plane. This real valued finite summation function is called kappa (K):

$$K(a, b, N) = \sum_{n=1}^N \left[\frac{\cos(b * \ln(n))}{n^a}, \frac{\sin(b * \ln(n))}{n^a} \right] \quad (2)$$

Here the function is a finite sum of N vectors in the plane depending upon the values of 'a' and 'b'. The value of 'a' is assumed to be 0.5 by the Riemann Hypothesis (RH), and the value of 'b' may be such that it yields a zero of the zeta function (that is, a solution to Equation 1 when it is set equal to zero) or not. As we'll see, using K to find Riemann zeros only requires an N no larger than b/π, so N becomes a function of 'b', returning the K function to just two variables: 'a' and 'b'.

Thus Equation 2 is the modified Riemann zeta function this paper explores. As vectors each term has a length of 1/n^a and an angle to the x axis of b*ln(n). Unless specifically noted otherwise 'a' is set to 0.5 in the following calculations and images.

The Static Geometrical Images and Definitions:

The programs written to explore this new finite real-valued kappa function have produced many different images of the function to demonstrate various characteristics of the mathematics that generate these images. There are basically just three major types of images, though each type may have many variations. In this section we display the geometric images and define attributes of those structures. In a following section we will look at the mathematics of these geometries.

The first image we look at is the 'n-form structure' which consists of placing the first term of the sequence as a vector starting at the origin and going to the address generated by Equation 2 for $n=1$. In this example we use a value of 'b' that is a known zero: $b = 60000.4412\dots$, and $a = 0.5$. Thus, the first vector lies along the x axis with length 1.0. We then proceed to draw the next vector (term $n=2$ of the sequence) starting at the head of the previous vector and at an angle of $b \cdot \ln(2)$ to the x axis and length of $1/\sqrt{2}$. Continuing this process for increasing 'n' values yields the beautiful image displayed in Image 1. (All images shown at end of paper, bound separately for ease of reference.)

We can now begin to define characteristics of this structure, which we call the 60k n-form. First, the structure is made of segments (the vectors) the first of which has a length of one. Proceeding around the structure counter-clockwise the segments become increasingly shorter, so much so that after a few hundred or so they appear as dots in the image.

There are many spirals in the structure and each spiral goes inwards in a clockwise manner, turns around at a spiral inflection, and comes out in a counter-clockwise manner. There is a second type of inflection halfway between each spiral called the medial inflection, where the curve changes from curving to the left to curving to the right as 'n' increases. With only one exception the two types of inflections alternate with increasing 'n' values.

Notice that the last spiral of the structure is centered on the origin. This only happens when 'b' (and 'a' of course) forms a zero of Equation 2, which also means it is a zero of the original Riemann zeta. Next notice the small section shown in red. This is the saddle and always appears in some form of every structure (of large enough N), zero or not. The exception noted above is at the saddle. In the red section of the saddle in Image 1 are two medial inflections without an intervening spiral inflection.

There are many symmetries in this structure, some of which are shown in Image 2. A line drawn from the origin to the center of the saddle, called the center line or the reflection line, bisects the structure, but only for zero values of 'b'. The center line is the *perpendicular bisector* of lines connecting the center of spirals to the left of the center line to segment vertices on the right. These lines are called spiral-to-segment lines, as shown in Image 2. Not shown is the characteristic that lines connecting spiral centers correspond to the segment lines reflected across the center line.

Image 3 shows a structure of much smaller zero value: $b = 48.0051\dots$, $a = 0.5$, and $N = 800$. There are three things demonstrated here. First, there are too few segments to find the saddle. Second the spiral inflection is where the segments change from angling to the right to angling to the left and those two segments pass very close to the origin. And third, the spiral inflection is the last inflection of the structure. There can be no more inflections of either type and the last spiral continues outward forever. It is not necessary to show terms beyond the last inflection.

An important definition concerns the last spiral inflection: it is the convergence of the kappa function. But since the function is *not* convergent we call it a quasi-convergent function. No matter what values of 'a' and 'b' are selected, all kappa functions, zero or not, are quasi-convergent. A quasi-convergent function is one that has a last spiral and the center of that spiral is its 'convergent point'. Defining the 'center of the spiral' can be a little tricky as we'll see shortly.

The second type of image, the angle-form structure, takes the same vectors formed from the terms of the kappa function as the n-form but assembles the structure in order of each vector's angle from the x axis. (A small variation, explained later, is that the first vector, of length 1, is placed last rather than first.) This yields a surprising but quite logical image for 'b' values that are zeros. Since the segments are vectors and the sum of a finite number of vectors does not depend on the order of the summation, the angle-form structure also indicates zeros by returning to the origin, as Image 4 demonstrates using the 60k angle-form structure. (The scale shown of the angle-form is 1/20 that of the n-form.)

Image 5 superimposes a green circle over the the 60k angle-form structure to demonstrate how closely, but not perfectly, the structure approximates a circle. The circumference and radius of this circle will play an important part when we explore the mathematics behind the structures in a later section of this paper.

The third major type of image is the vector display. This simply displays all the vectors 1 through N with their base on the origin. Each vector (segment) is length $1/\sqrt{n}$ and at an angle from the x axis of $b \cdot \ln(n)$. Image 6 shows this type for the 60k structure. First vector (n=1) is 1 unit in length and lies along the positive x axis.

Before leaving this section some images of n-form structures that are zeros and not are shown in Images 7 - 10. Not all images are as uncluttered as 60k! (In Image 8 the saddle almost touches the origin.)

The Dynamic Geometrical Images and Definitions:

The previous images have all been formed with values for 'a', 'b', and 'N', that are fixed. Quite a bit more can be learned by observing what happens as one or more of these parameters is varied.

Increasing 'N' simply adds more segments to the structure. Eventually there are enough terms to reach the last spiral inflection which is dependent only on the value of 'b'. After that more terms (more segments) just adds to the outward bound last spiral. Since the length of the segments is dependent only on the term number ('n') the lengths of the vectors are the same for any value of 'b'.

Changing 'b' on the other hand dramatically changes the structure. Image 11 is a composite of different 'b' values starting with the 60k n-form. The first six images increment 'b' by about 0.1 each. Notice that the last spiral moves counter-clockwise around the origin, twisting in on itself until it lands again on the origin indicating a new zero value of 'b'. As we continue to increase 'b' by more and more the structure twists into a tangled ball finding a new zero with each full revolution of the last spiral. After awhile (at around $b=60035$) it starts to straighten out making very large counter-clockwise circles. The last image in this set shows a composite of six structures as the last spiral circles around to find a zero before again tying itself into knots.

We can track the last spiral inflection showing its path as it rotates counter-clockwise. Image 12 shows that path from $b=60000$ to $b=60040$. Each time the path goes through the origin it reveals another zero. This image closely resembles a similar image generated by the actual Riemann zeta.

What about changing the 'a' value? After all, isn't that what the RH is all about? Since the 'a' value controls only the length of the segments and has nothing to do with the angles of the segments changing 'a' just magnifies or reduces the size of the structure but this change of 'a' does move the last spiral away from the origin, so it no longer finds zeros. Image 13 shows multiple values of 'a' from about 1.3 to about 0.3. The structure is the 60k n-form, and the 14th structure in from the largest is the $a=0.5$ zero. Empirically the last spiral inflection never returns to the origin with any value of 'a' except

0.5. However, since this is the modified zeta function, every value of 'a' produces a structure. There are no poles - even $a=0$ yields a structure.

The Mathematics of the Geometry:

Now that we have images of the kappa function and definitions of many of the characteristics we can start to answer specific questions such as, 'just what causes an inflection', and 'why is there a last inflection'? Again we'll have the aid of many images provided by JavaFX programs written for that purpose, but this time we'll try to explain the mathematics that causes the structures to look as they do.

The kappa function at its most basic level is a summation of a finite number of terms. Each term generates two numbers from the values of 'a', 'b', and 'n': the x and y coordinates of a point in Cartesian 2-space. Image 14 shows most of the terms' points for the 60k structure. Not much information available there. Viewed as vectors from the origin we have Image 6, again not much help. However, the function is a summation, so viewing the addresses as those of vectors but adding them together (vector addition) we have the structure of Image 1. Two very important characteristics of this structure are the medial and spiral inflections, but how do we know how large to make N to include all of the inflections? And how do we use these structures to locate the Riemann zeros?

As an aside, a feature (flaw?) of modern mathematics is that 'pretty close' is 'close enough' for many calculations involving infinite sums or calculus, as two examples. Part of the problem is that we use irrational numbers as if they were well defined. We know exactly what 3.1 is, but neither we as humans, or machines (as whatever they are) can actually know or use π in calculations. We use an approximation taken to whatever precision we need. We can not, nor can machines, add all the terms of an infinite sum, but we can use an approximation, or define a 'limiting value'. Since the Riemann zeros are likely to be irrational numbers even after we 'find' them we cannot completely express them. So like it or not, 'close enough', is, well, close enough.

Again thinking of the segments as vectors with length, and direction expressed as their angle with the x axis we realize that each vector (except the first) has a predecessor and the difference between their two angles is their relative angle (also the external angle). The angle θ of each vector is determined by the value of 'b' and of 'n': $\theta = b \cdot \ln(n)$. For a given structure with a fixed 'b', each vector's θ will be fixed but the difference between vector 'n' and vector 'n+1' will cycle between 2π and 0 as 'n' increases. When the angle between two adjacent vectors is 0 they are parallel and point in the same direction. This is what a medial inflection is: when as you move up the vector chain the angle between successive vectors passes through 0. Image 15 shows the last two spirals of 60k n-form. In red are vectors with term numbers (n) 9547 through 9551. One of those vectors will have an angle slightly less than its predecessor and the following vector will be much greater having cycled through the modulus 2π and have an angle almost 2π greater than its predecessor. In fact, vector 9550 differs from vector 9549 by just 0.00057 radians while vector 4951 differs from vector 4950 by 6.2832 radians. So vector 4950 is a medial inflection vector.

Between every two spirals there occurs this medial inflection caused by the changing vector angles. In the heart of every spiral a spiral inflection occurs for a similar reason. In the spiral inflection scenario the difference in angles between two successive vectors passes thru π ; the vectors are close to parallel but opposite in direction. Image 16 shows vectors 7, 8, and 9 of a structure with $a=0.5$, $b=25.01085758$, and $N=16$, which is a zero. Notice that vector 6 turns right into vector 7; 7 turns right into 8; but 8 turns left into 9. Vector 8 is the inflection vector and its midpoint passes very close to the origin indicating that this 'b' value is a zero.

But how close is close enough to indicate a zero? And why does a vector's midpoint indicate a zero instead of one of its endpoints? The definition of a quasi-convergent function (or series) is designed to

answer these types of questions. First we must look very closely at the center of a spiral. Image 17 is our old friend 60k n-form but enlarged 100,000 times. For this structure N=13098 instead of the 19098 needed to reach the center of the last spiral. This is so the detail of the vectors is not completely covered by the vectors themselves. Notice how vague polygon shapes can be seen. These are made by the vectors as they spiral in. Each vector is a little bit shorter than its predecessor and makes a little bit sharper angle to its predecessor, slowly changing a hexagon to a smaller pentagon to a smaller square, etc. The origin is ‘obviously’ in the very center of the spiral but what mathematical way can we use to find it?

Image 18 is made with N=17098 and the inner circle is getting smaller. Very importantly, the circle is not made up of vectors laid end to end in a circle. Rather, the vectors are much longer than the diameter of the circle and only tangentially form the circle.

Image 19 is finally the full N=19098 terms to the last spiral inflection and we’ve had to enlarge it 10,000,000 times. The final vectors are going all the way across what’s left of the inner circle. The midpoint of the 19098th vector (just to the left of the slightly larger gap) is about 2.25E-7 (0.000000225) from the origin. That is close enough. The value of ‘b’ here is 60000.441207279. Changing ‘b’ (or ‘a’ for that matter) by a very small amount completely destroys this image. Setting b=60000.44121, a change of 3 millionths, results in Image 20. The red axis lines indicate the origin.

Additional terms (increasing N) starts the vectors unwinding after the last spiral inflection forming a very dense ball of vectors.

How did we know that 19098 terms were needed to find the last spiral inflection? Empirically discovered formulas have guided us to many of the salient features of the structure. One of the most important of these formulas is $N=b/\pi$. That is, the number of terms needed to reach the last inflection, N, is just the value of ‘b’ divided by π . Why? Good question, but as yet an unanswered question, as is the case for most of the formulaic expressions that follow. A latter section of this paper demonstrates why there is a last inflection.

real irrational number:	nearest integer number:
b / π	N, last spiral inflection, last inflection
$b / 2\pi$	last medial inflection
$b / 3\pi$	2 nd last spiral inflection
$b / 4\pi$	2 nd last medial inflection
$b / (2k-1)\pi; 1 \leq k$	spiral inflections
$b / 2k\pi; 1 \leq k$	medial inflections

From this table we can see that half of the total number of vectors are in the last spiral. Another formula of empirical nature: the center of the saddle is at $\sqrt{b/(2\pi)}$.

Equation 2 can now be re-written to reflect $N = b/\pi$:

$$K(a, b) = \sum_{n=1}^{b/\pi} \left[\frac{\cos(b * \ln(n))}{n^a}, \frac{\sin(b * \ln(n))}{n^a} \right] \quad (3)$$

Another as yet unexplained characteristic of this function concerns the sigma notation. Convention expects the starting value of 'n' to be 1, or sometimes 0, and that the value of 'n' be incremented by 1 until the upper limit is reached. Thus far all our images and calculations follow that convention. However, it has been discovered that one can start with any positive real number k and increment by that same number k and only effect the orientation and scale of the structure. For example we can start with n=1.5, next n=3.0, next 4.5, next 6, etc. continuing on until n=1.5*b/π and arrive at the same structure 2/3 sized and rotated about the origin by b*ln(1.5). The new structure has the same symmetries, the same angles between vectors, just shorter vectors, and most importantly, it still finds all the Riemann zeros.

What is even less understandable is that we can start with any positive real number and increment by twice that number and still find all the Riemann zeros, although the symmetries are lost. As yet no combinations other than 'start k increment k' and 'start k increment 2k' have been found.

An interesting characteristic of the angle values generated by the zeta functions arises from the way logarithms work: the sum of two logs is the log of the product. Take two term numbers, say 'j' and 'k'. The sum of the angles of term 'j' and term 'k' is equal to the angle of term j*k ('j' times 'k'). This was noticed for the kappa function by a fellow researcher while going through long lists of data, and is true without regard to the values of 'a' and 'b' or if those values are a zero of the function. This single observation may help to explain the above two paragraphs, and perhaps shed light on the RH.

Turning from the n-form structures to the angle-form images there are three issues to examine. First, to maintain as much symmetry as possible in the images the first vector, which is always length 1 and angle 0, is moved to the end of the list of vectors in angle order. This is just calling an angle of 0 an angle of 2π, but it makes for a prettier picture.

Second, when we draw a circle for comparison to the angle-form we calculate its radius and offset from the origin to match as best as possible the angle-form image. To do this we assume that the total length of all the vectors is the circumference of a circle. Dividing by 2π gives us the radius and the vertical offset for the center of the circle. This seems to work very well, despite the fact that the angle-form images are actually irregular polygons.

Third, it was empirically found that the sum of all the vector lengths was very close to 2√N (or from the above table, 2 times the square root of b/π). In fact, for values of 'b' greater than around 100 the difference between the 2√N and the sum of lengths hovers around 1.4, and at 'b' values around 100,000 it stays around 1.46, regardless of whether or not 'b' is a zero. Why this difference between the square root of the number of terms and the total length of the vectors remains a relative constant is a bit of a mystery, but the value is known as zeta(1/2) = 1.46035... (see <https://oeis.org/A059750>)

The final topic of this section answers the question of why there is a last spiral, which is what allows the subject of this paper, quasi-convergent functions, to exist at all. The sum of the vectors depends upon the length and the angle (to the x axis) of each vector. The lengths become shorter as 'n' increases and don't really impact the 'last spiral' argument. So looking only at the angles of each vector we know that the angle to the x axis is always b*ln(n), that is, the value of 'b' times the natural log of 'n'. Additionally we know that angles of any size are equivalently reduced to the range of 0 to 2π.

The angle between two consecutive vectors is what determines inflections: either when that angle passes through 0 (medial inflection) or through π (spiral inflection). The angle of a vector is b*ln(n) and the angle between two vectors is b*ln(n+1) - b*ln(n), or b*(ln(n+1)-ln(n)), which can be stated as b*ln((n+1)/n).

For a given 'b' there will eventually be an 'n' such that $\ln(n+1)-\ln(n)$ is less than $1/b$ and from that 'n' on the angle between the two vectors will be less than 1 but always positive, meaning it will not pass through 0 or π to form any more inflections.

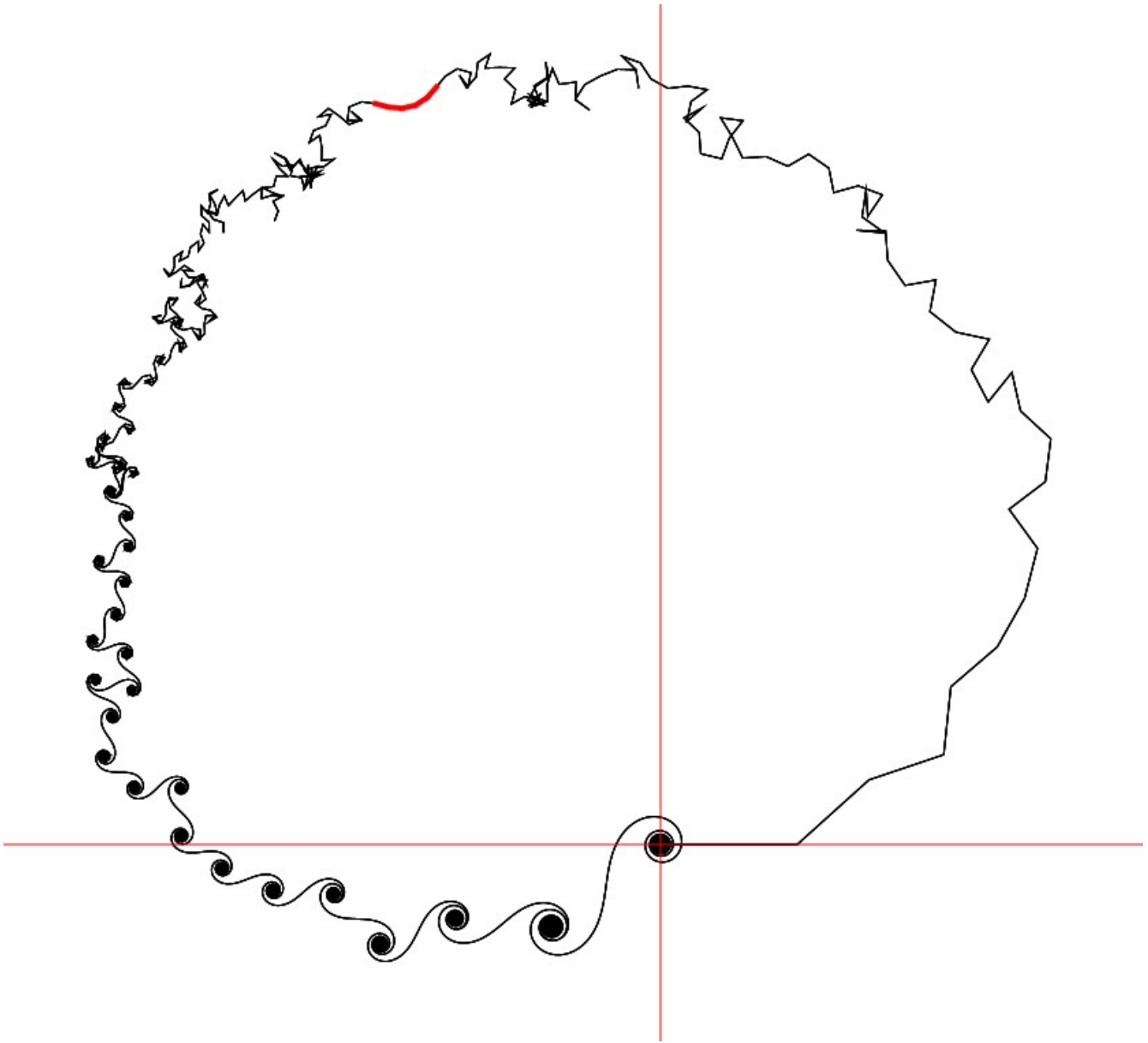


Image 1: $a = 0.5$, $b = 60000.4412\dots$, $N=19098$

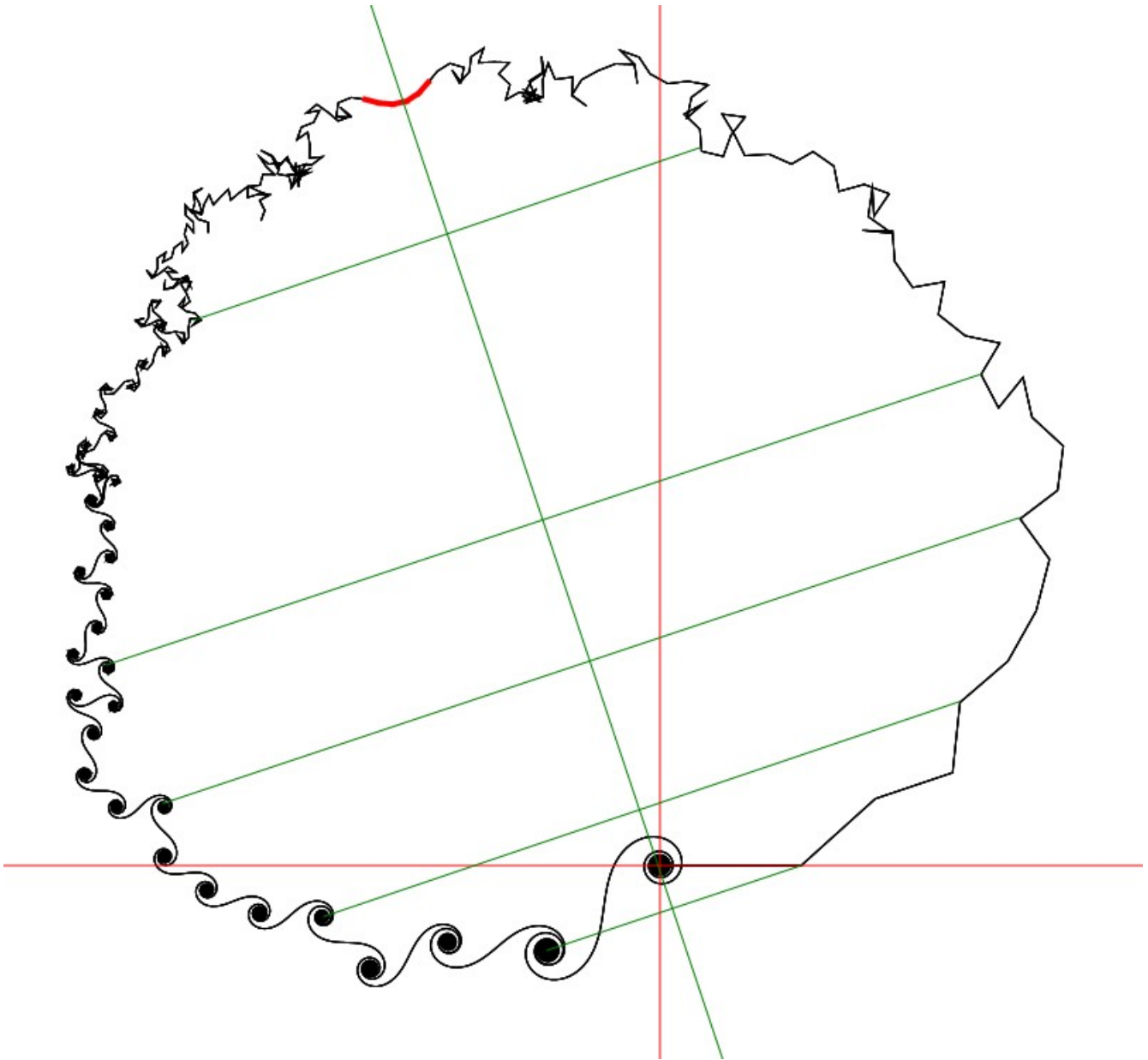


Image 2: 60k symmetries

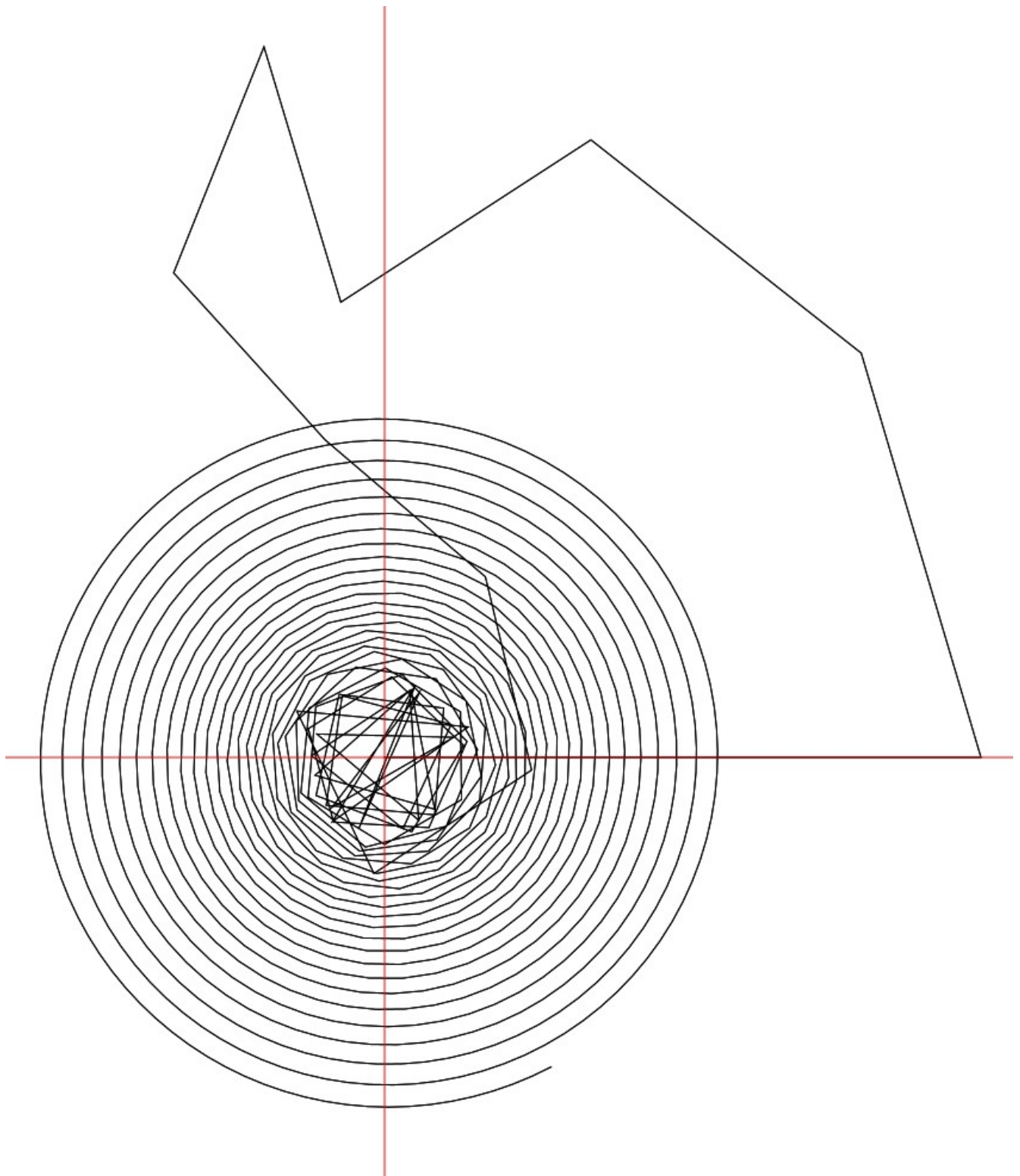


Image 3: $a=0.5$, $b=48.0051\dots$, $N=800$

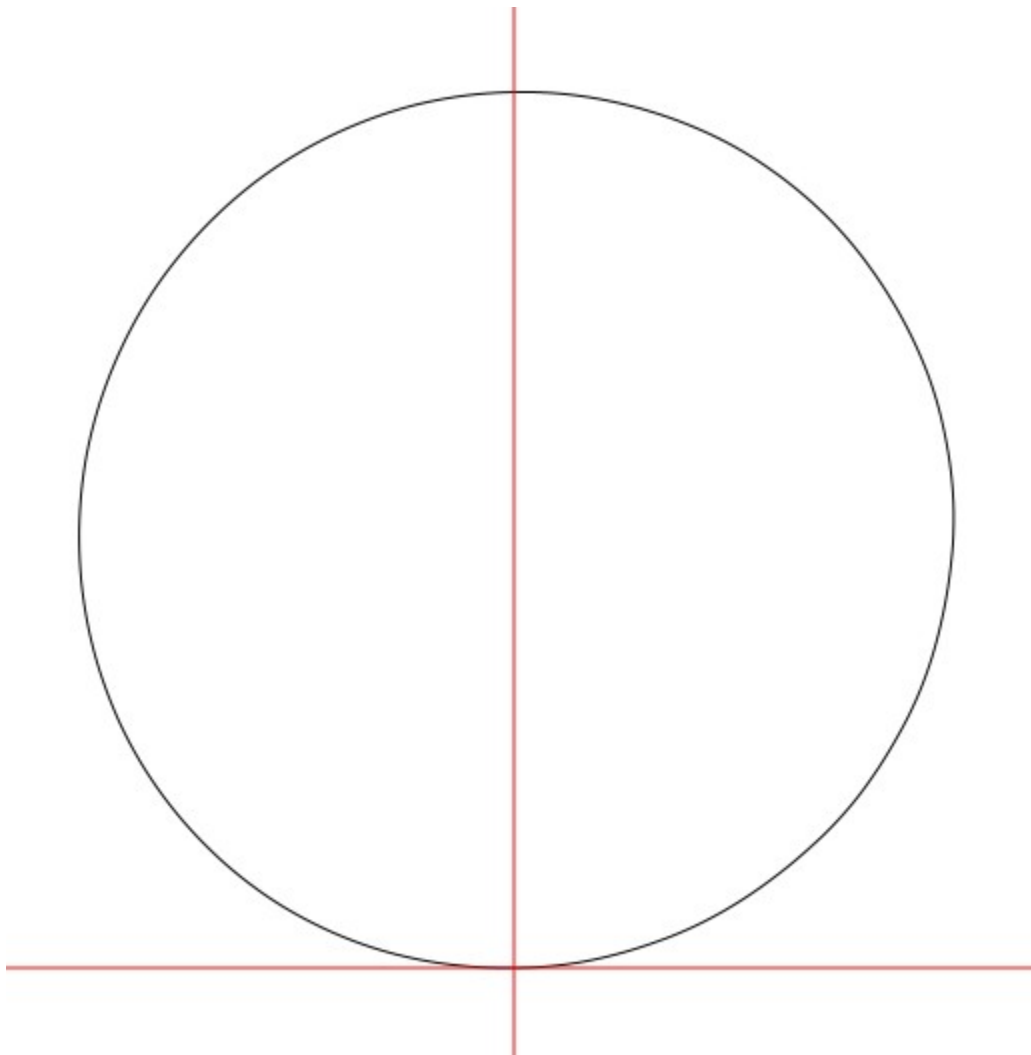


Image 4: 60k angle-form

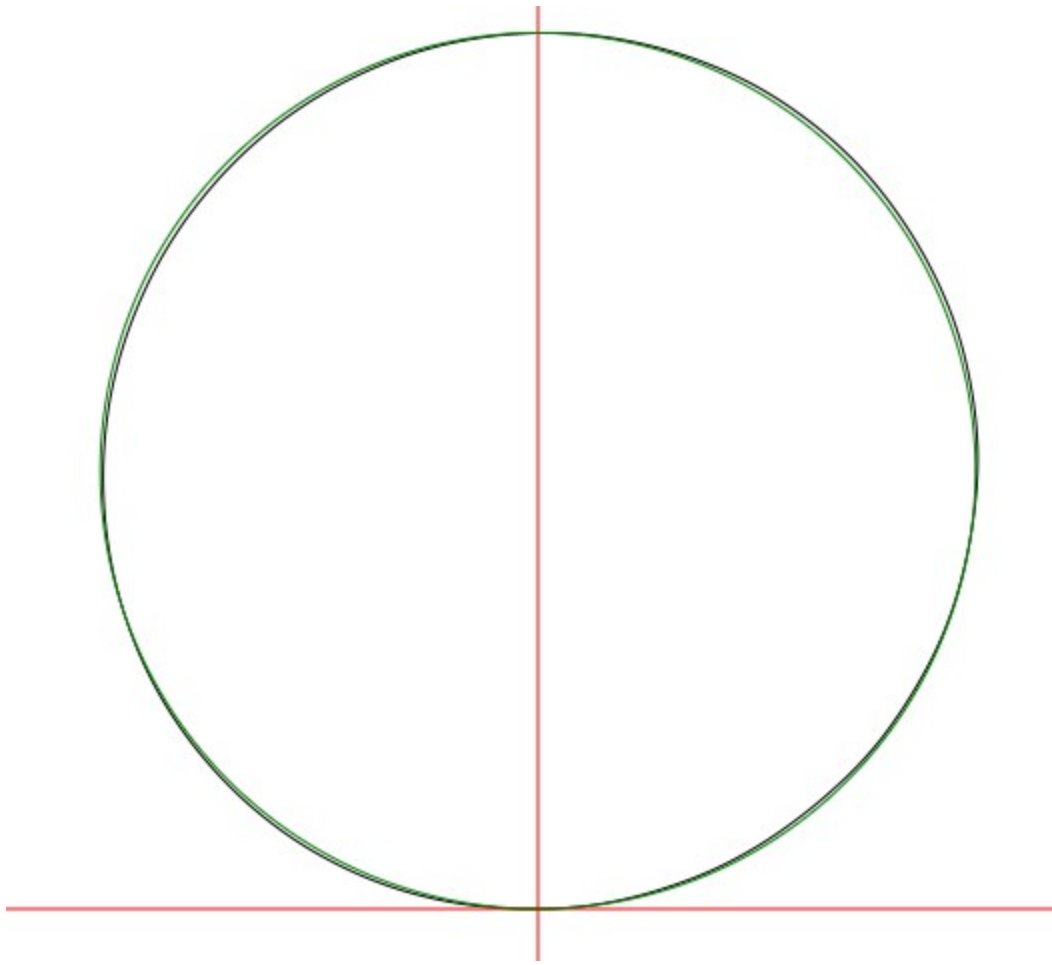


Image 5: green circle over 60k angle-form

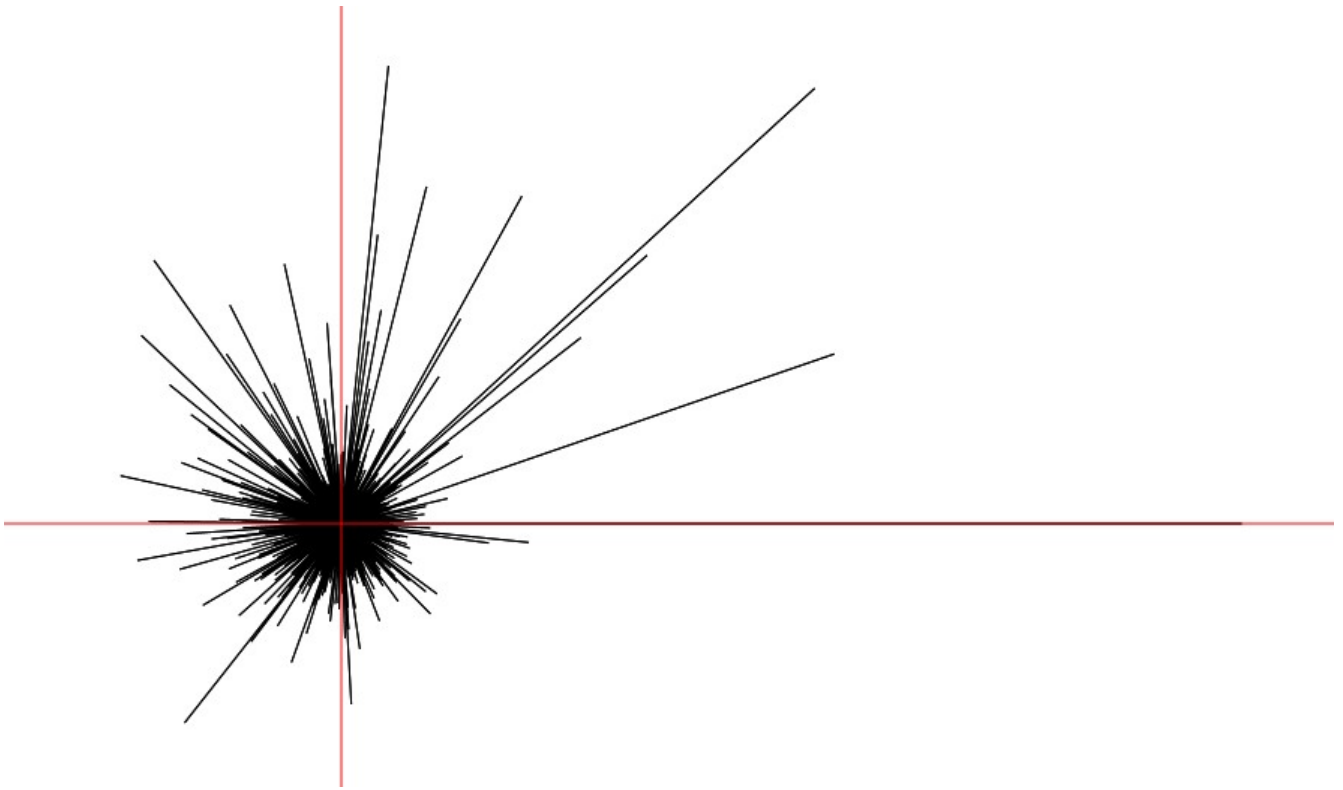


Image 6: 60k vector-form

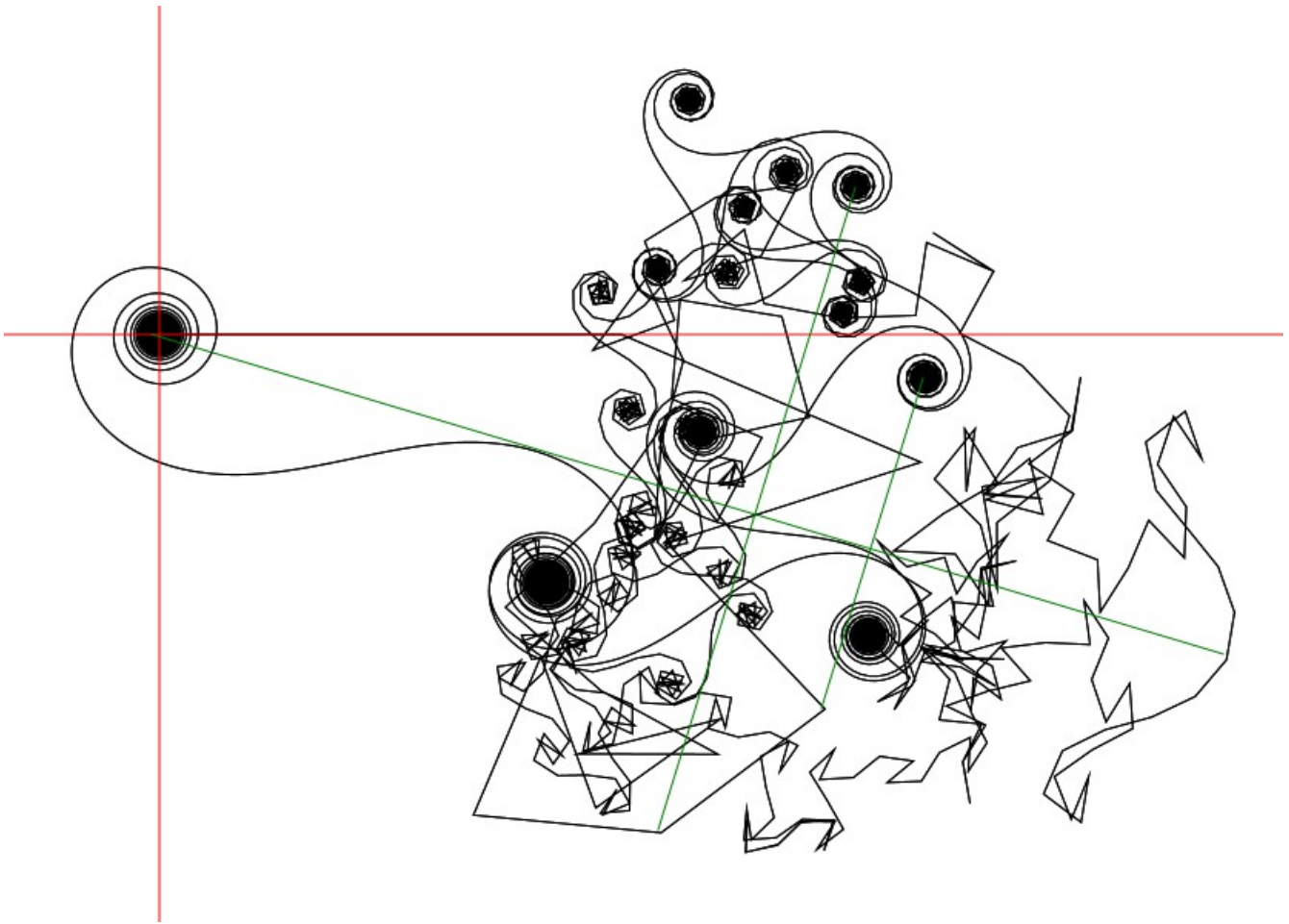


Image 7: $a=0.5$, $b=50000.4067\dots$, $N=15915$; a zero

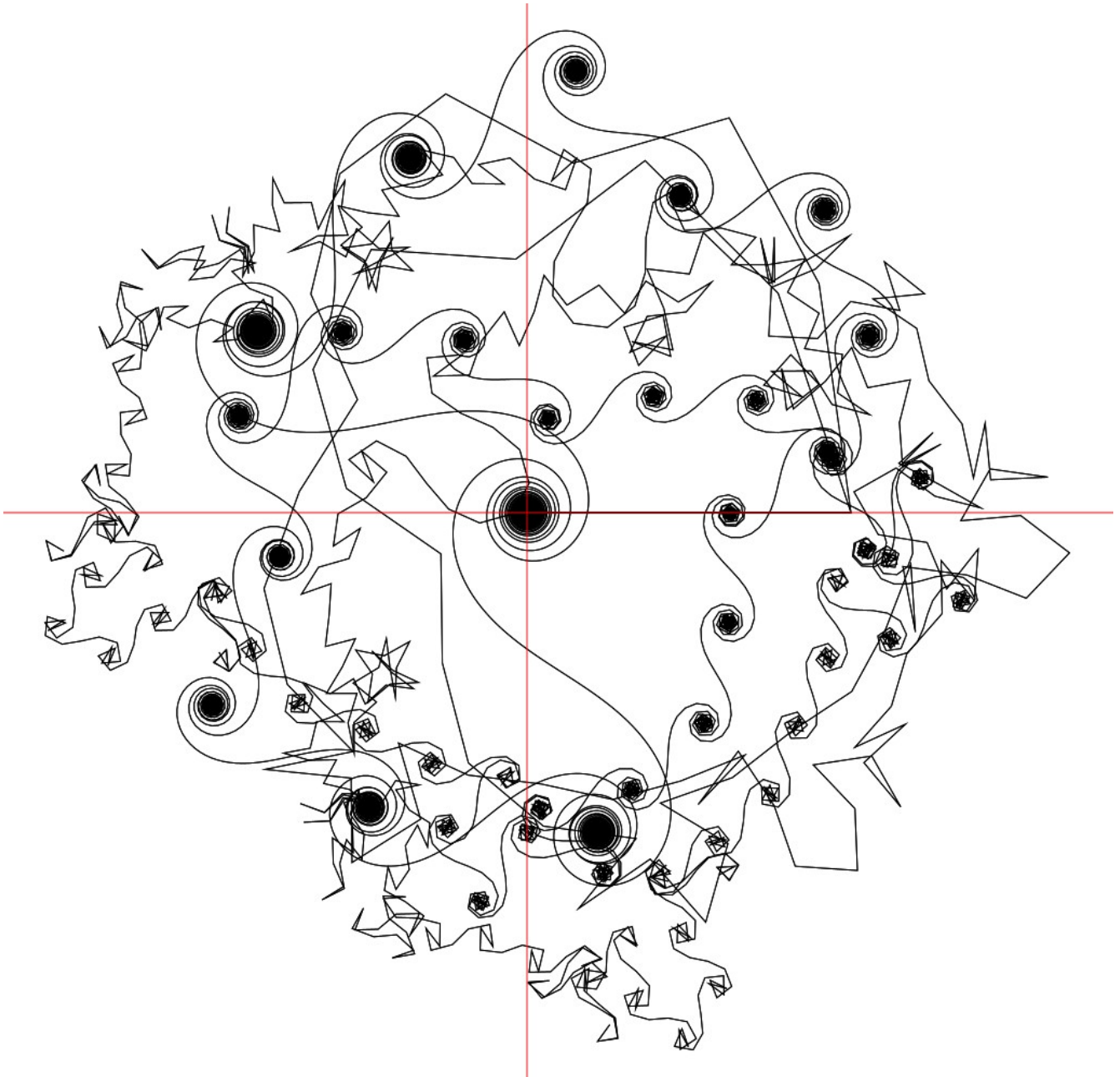


Image 8: $a=0.5$, $b=49976.5354\dots$, $N=15908$; a zero

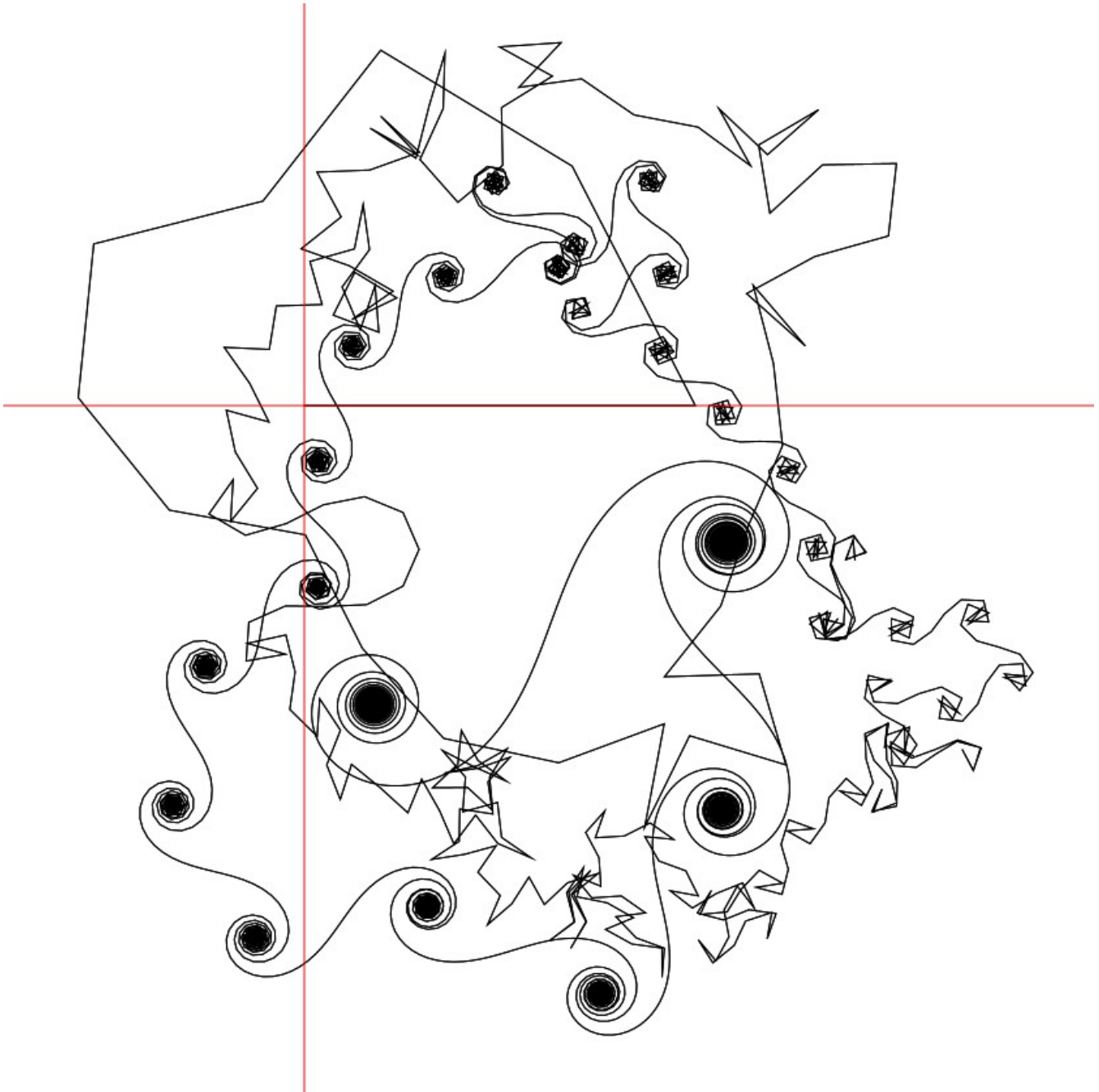
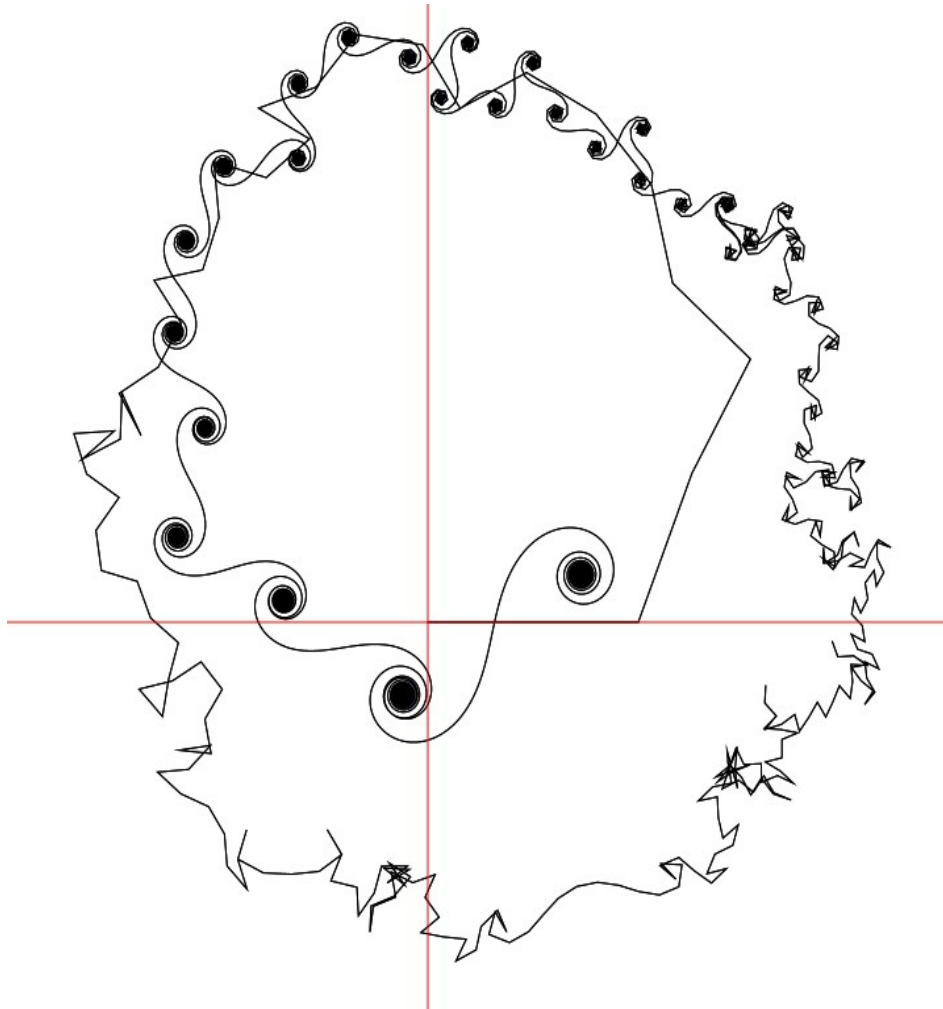


Image 9: $a=0.5, b=49976.7354\dots, N=15908$; **not a zero**



*Image 10: $a=0.5, b=60001.1175\dots, N=19098$; **not a zero***

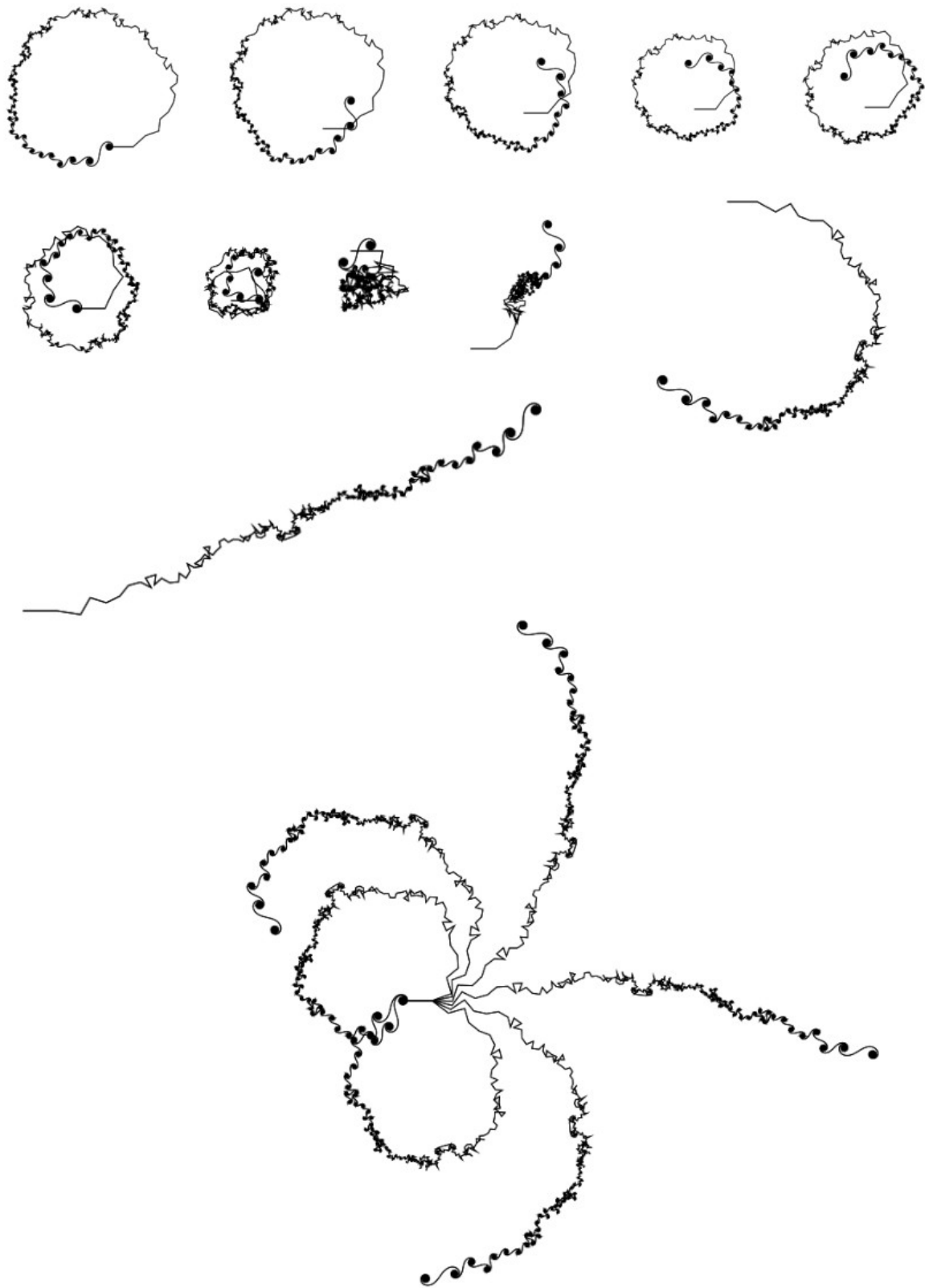


Image 11: changing the value of 'b'

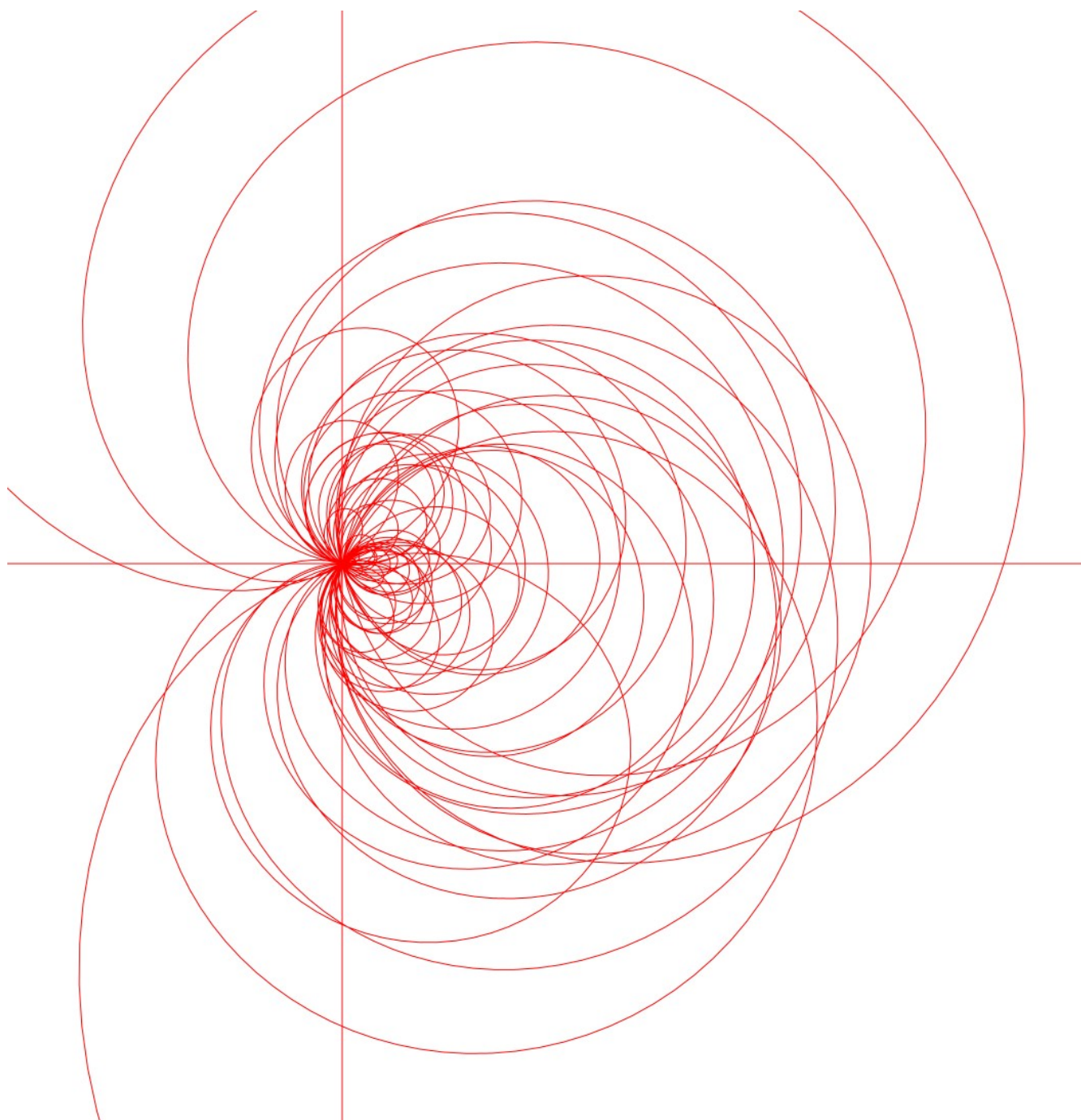


Image 12: track of last inflection with increasing 'b' value

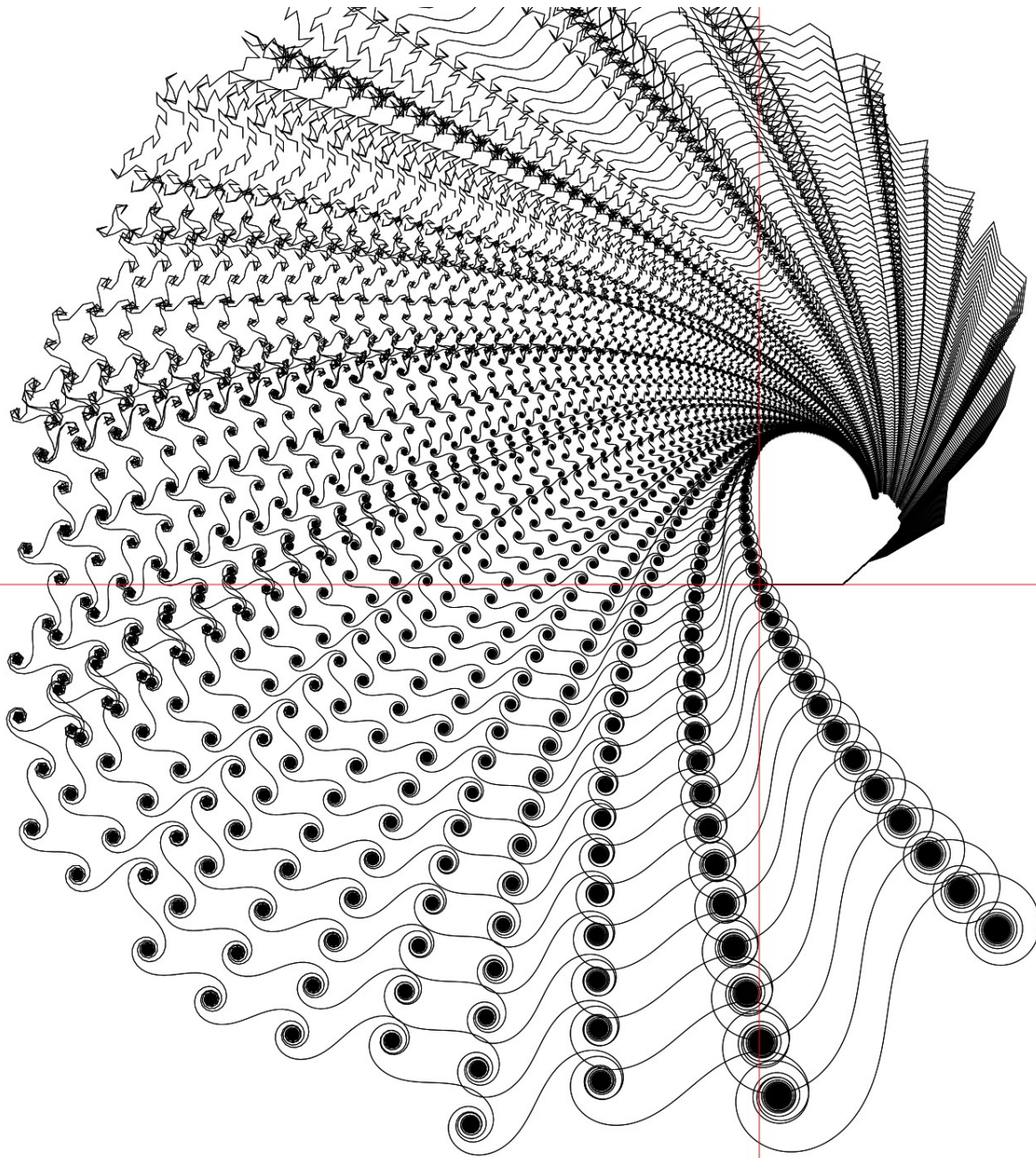


Image 13: 60k n-form multi 'a' values

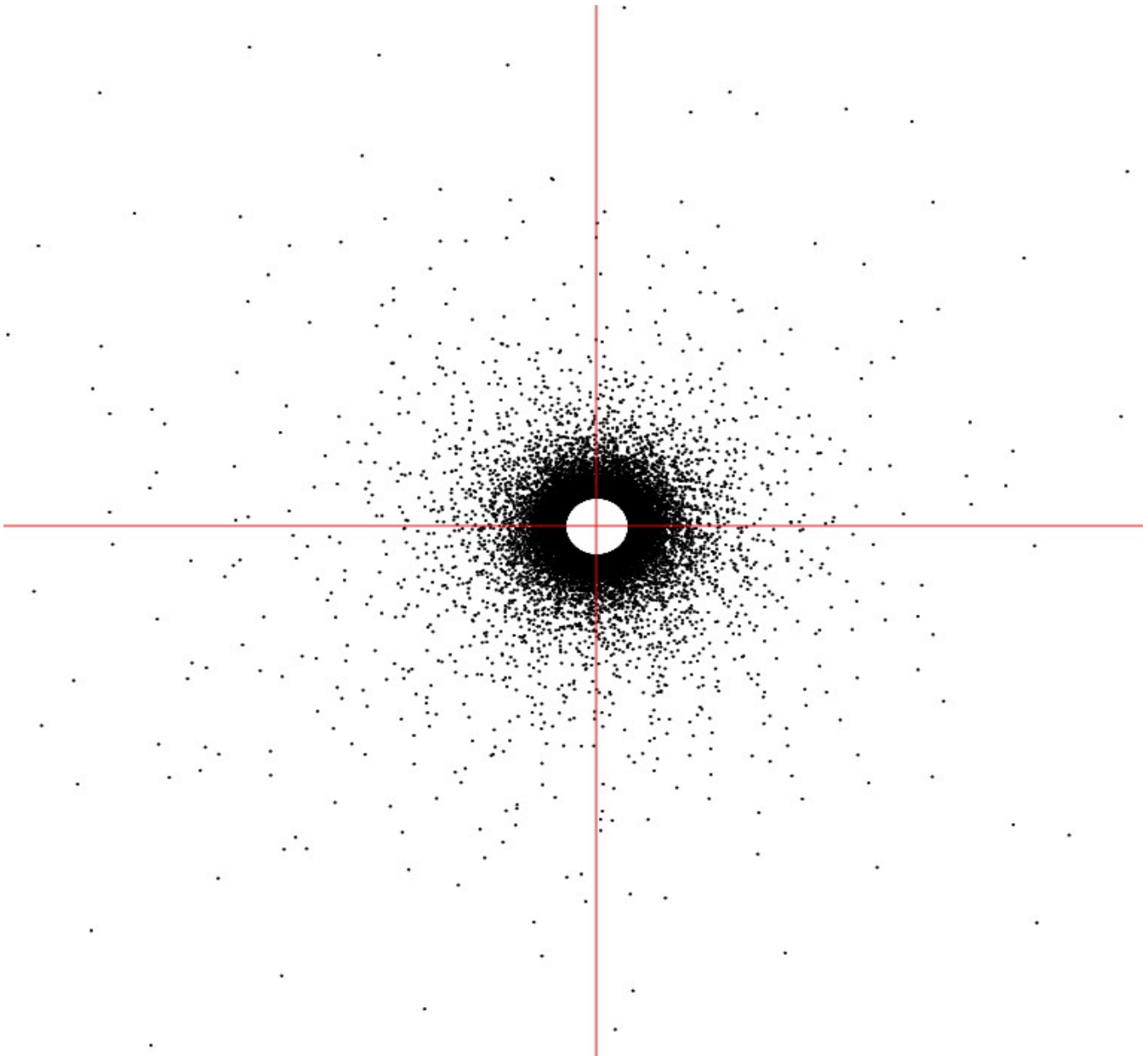


Image 14: term addresses for 60k

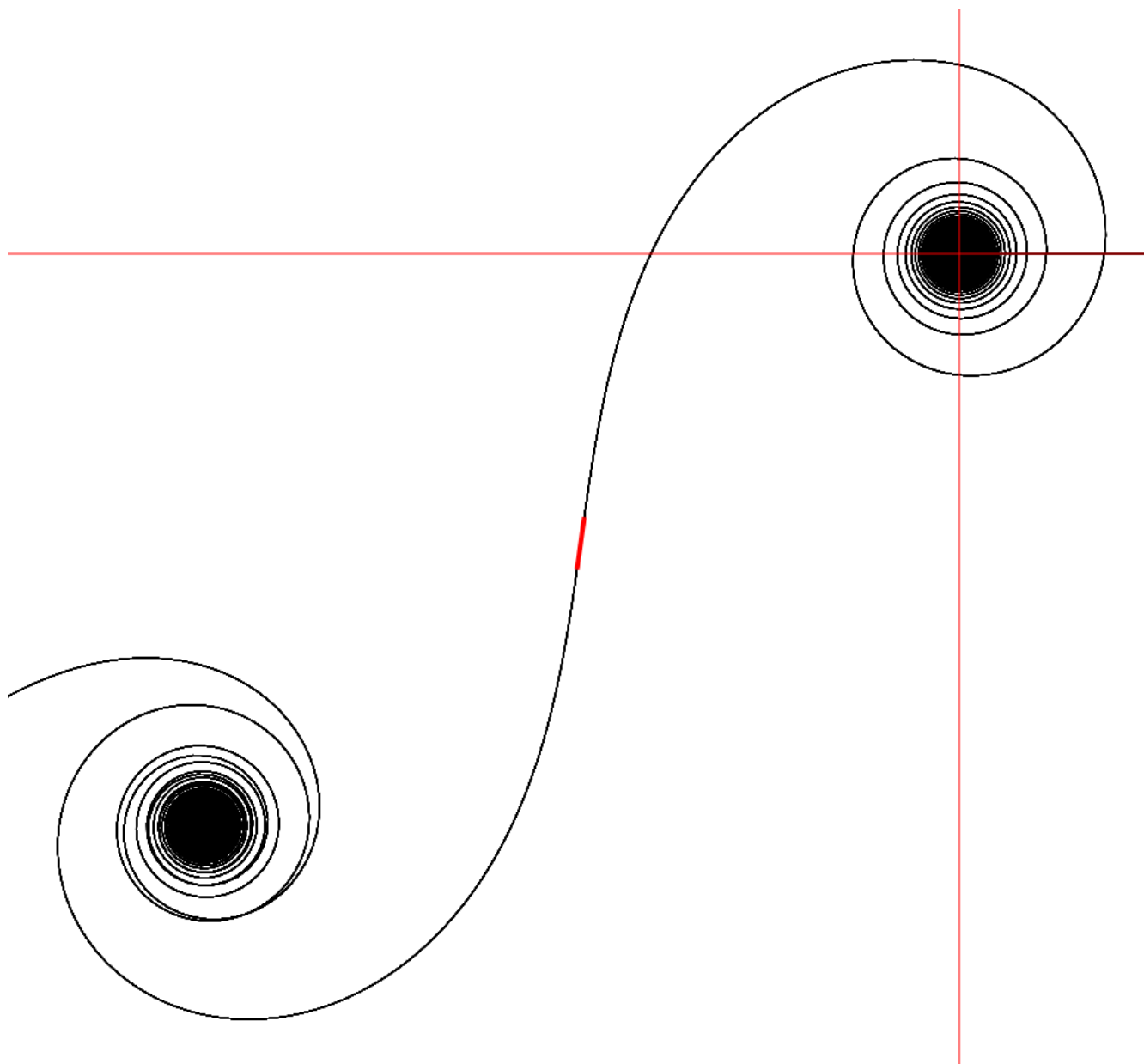


Image 15: medial inflection between last two spirals of 60k n-form

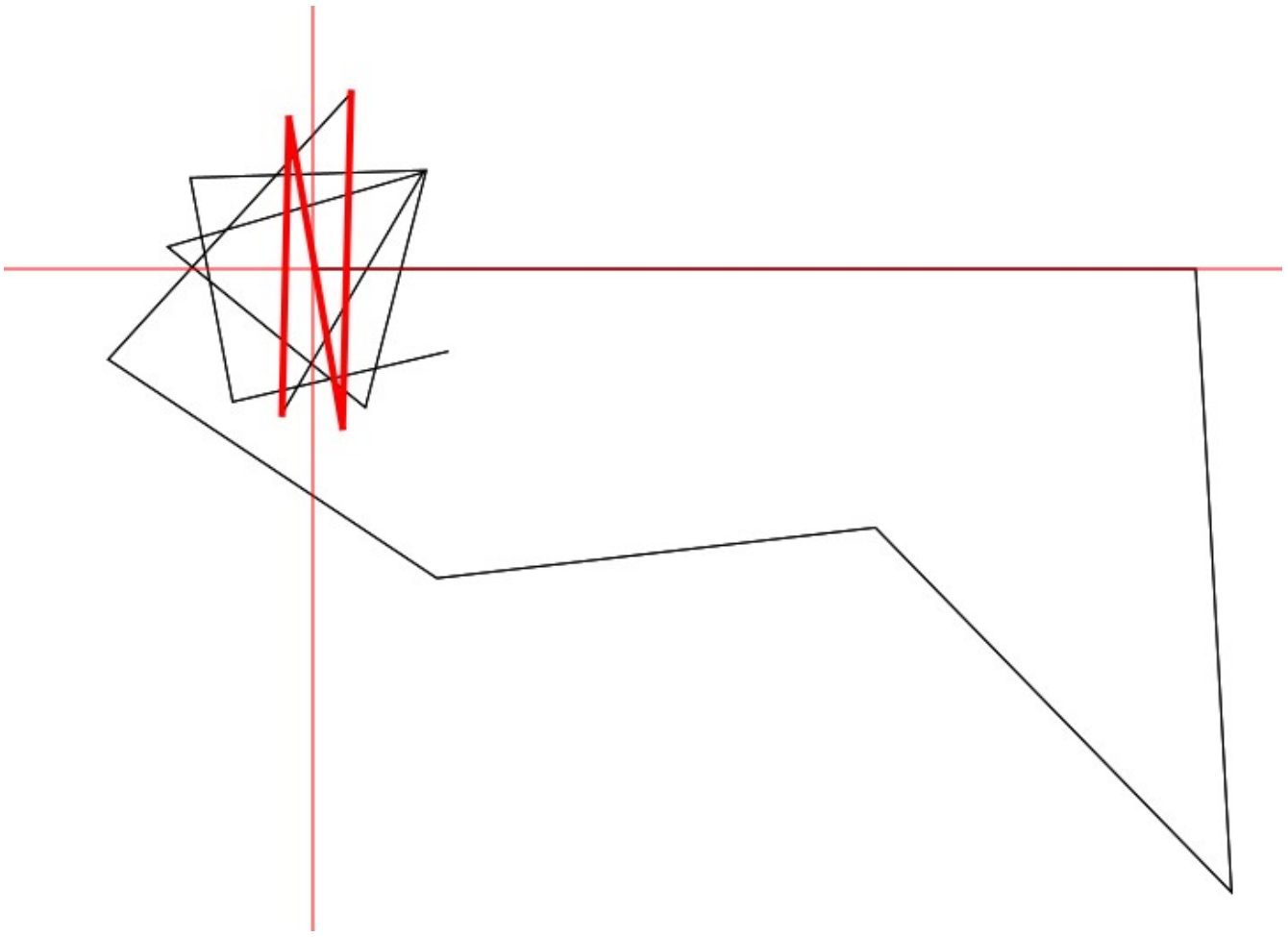


Image 16: $a=0.5$, $b=25.01\dots$, $N=16$; a spiral inflection

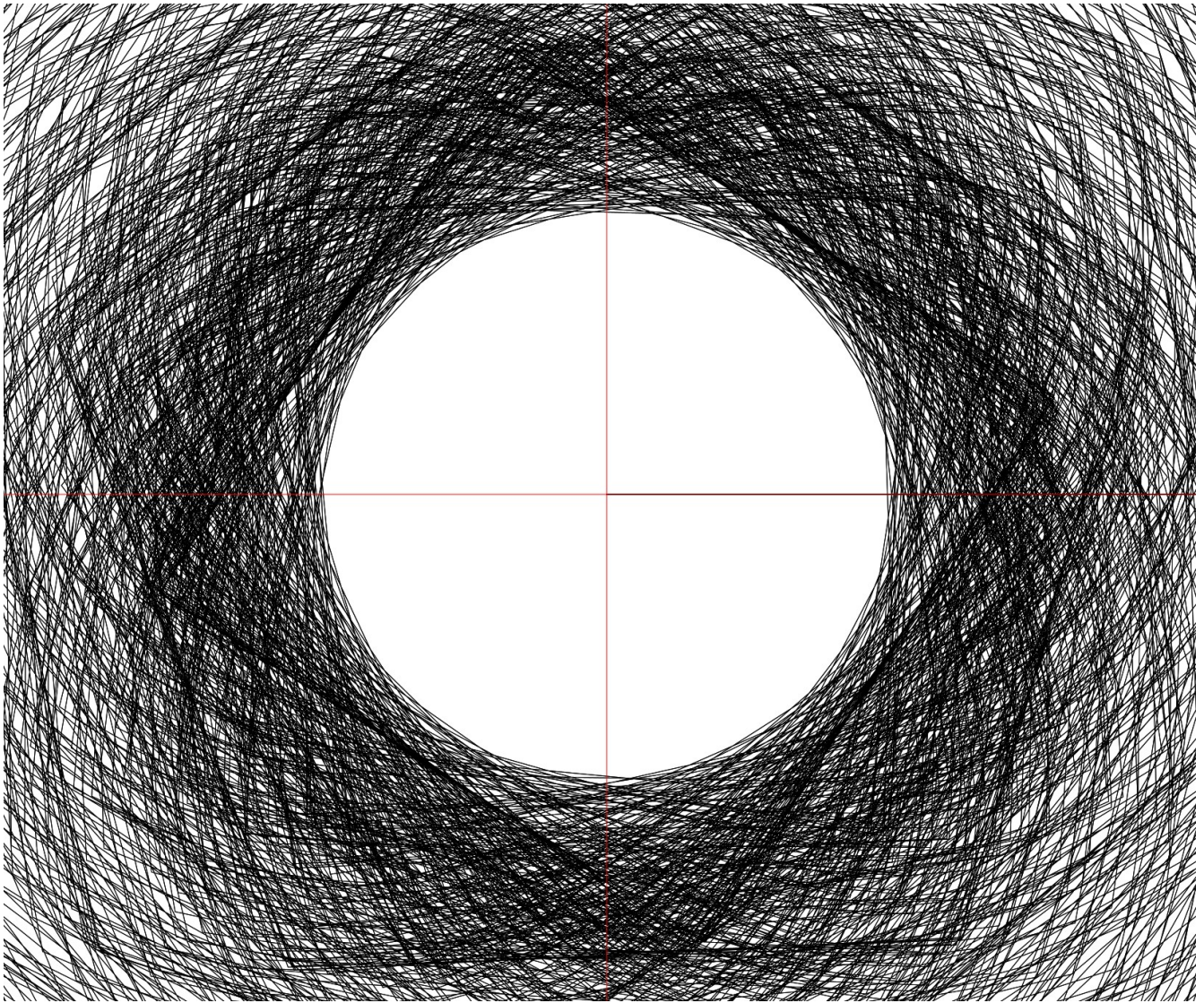


Image 17: 60k n-form enlarged 100,000 times, $N=13098$

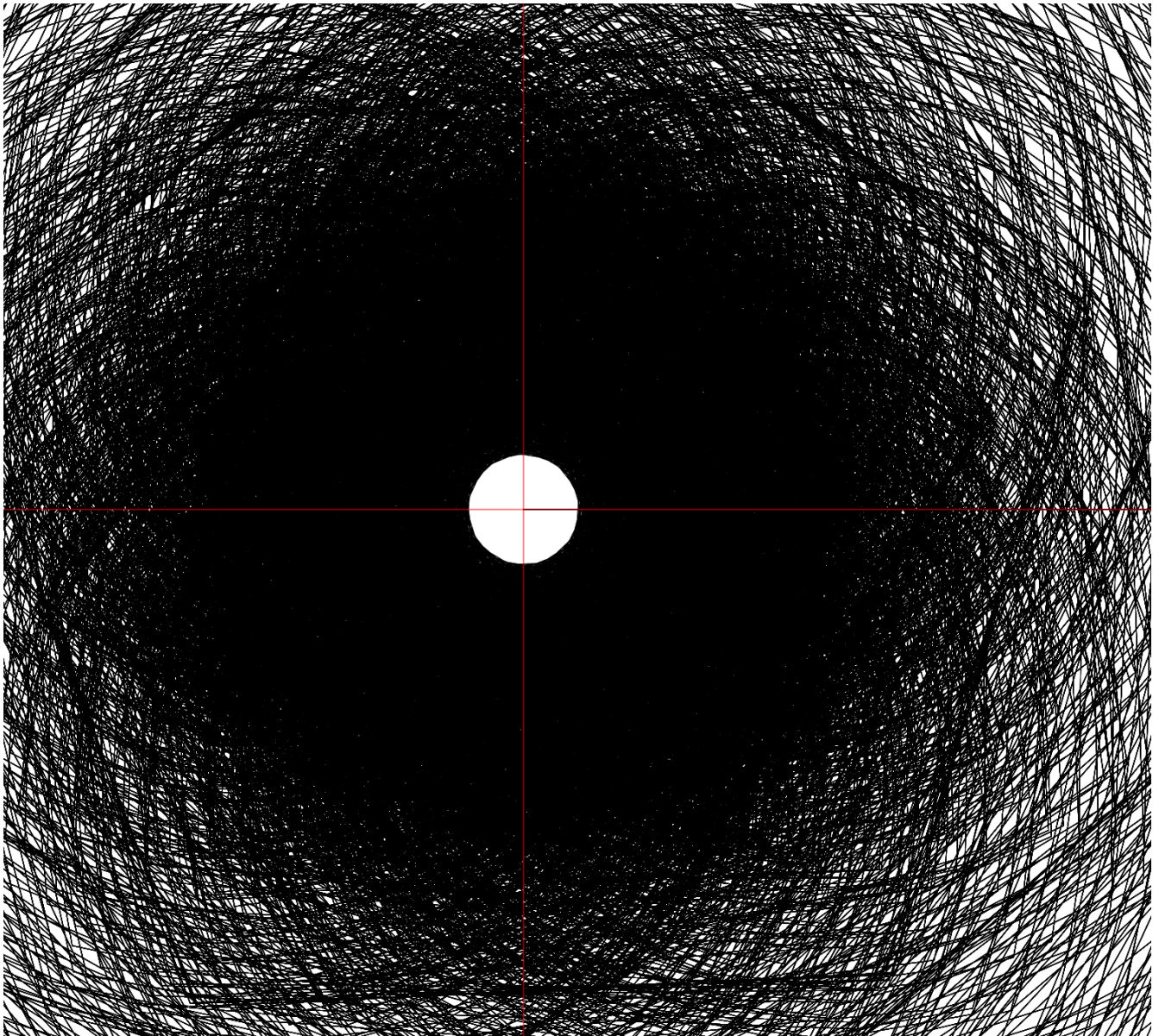


Image 18: 60k n-form N=17098

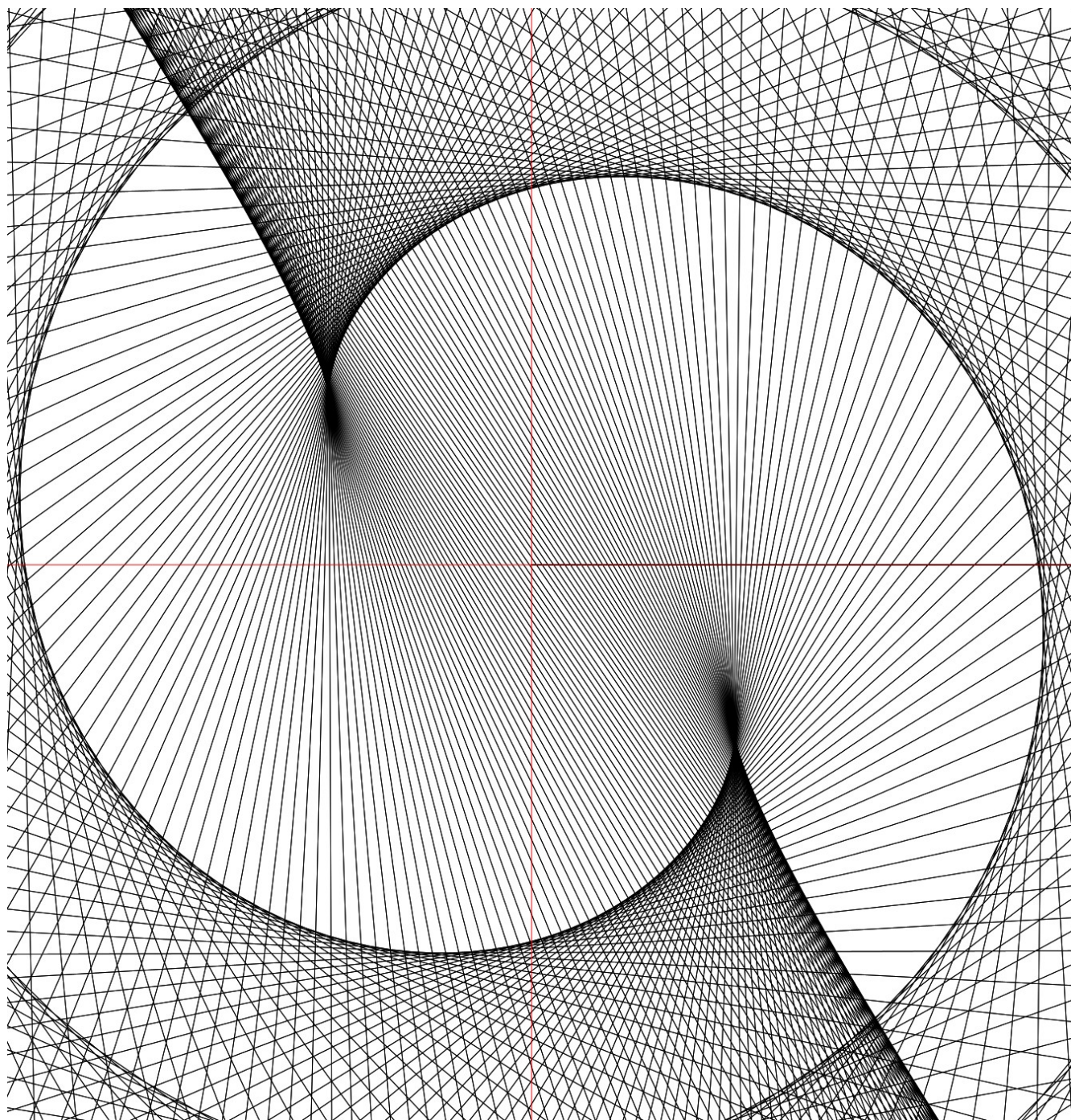


Image 19: 60k n-form $N=19098$ enlarged 10,000,000 times

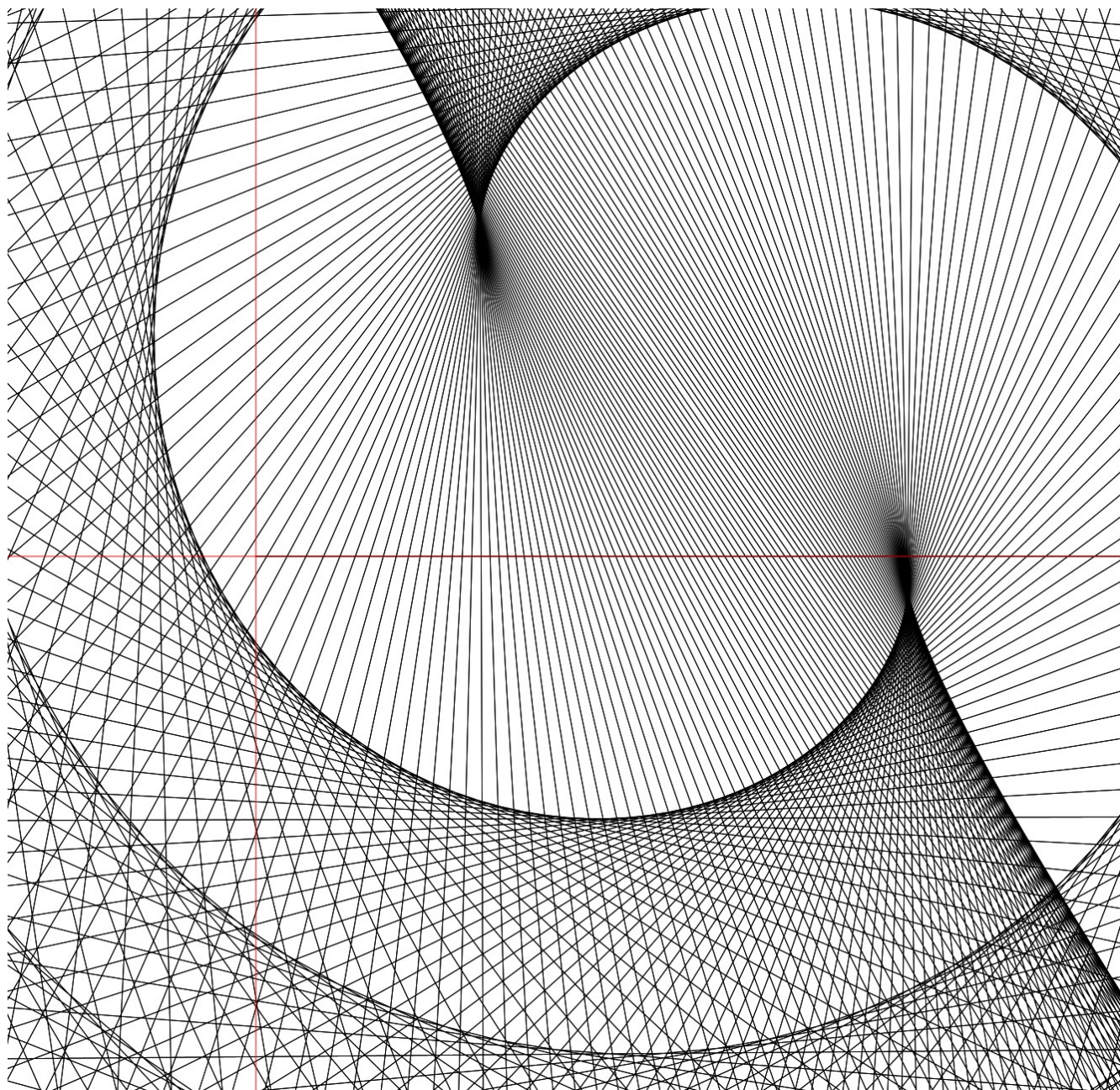


Image 20: not quite 60k n-form $N=19098$ no longer a zero