Developing a Quantum Calculus:
The Difference Path to Bernoulli Numbers

Calculus, also called infinitesimal calculus, is the analysis of continuous change. Standard discrete calculus, despite its name, is the analysis of continuous change approximated by Riemann sums of various intervals of the (continuous) independent variable. If the interval used in discrete calculus is allowed to approach zero discrete calculus becomes infinitesimal calculus.

Quantum calculus (my name for these operations) is similar to discrete calculus but its purpose and process are quite different. As science pushes into both smaller and larger realms of the universe it is becoming apparent that at least some (if not all) ‘things’ have a smallest amount. That is, things are made up of one or more discrete and indivisible bits. Even space and time may be as discrete as a bar of gold, which can only be divided into atoms of gold; beyond atoms it can not be divided and still retain ‘goldness’. Quantum calculus does not approximate a continuous function, rather, it defines a calculus designed to quantify discrete functions and processes – specifically, quantify discrete processes measured in indivisible units. Quantum calculus defines the h of infinitesimal calculus to be 1.

It is not too surprising to find that the Bernoulli numbers appear in the generation of quantum integration, but before we rediscover Bernoulli numbers we must define quantum differentiation.

The starting point of any calculus is rate of change of one measurable quantity against the change of another, called differentiation. Standard nomenclature has a function f applied to an independent variable x that yields a value of the dependent variable y, expressed as $y = f(x)$. As x changes so might y depending on the function f. A new function can perhaps be found that expresses the rate of this change, and finding this new function is the purpose of differential calculus. Infinitesimal calculus defines differentiation with this equation:

$$\frac{dy}{dx} = \lim h \rightarrow 0 \quad \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

Quantum calculus simply replaces h with the number 1 and does away with any application of a limit. This yields the well known forward difference equation:

$$f(x+1) - f(x)$$

for finding the quantum differential of the function f. There are difficulties with indices using this formulation, however, so, instead of forward difference, quantum calculus uses backward difference which also gives an appropriate modified form of Pascal’s triangle, useful later. The equation for quantum differentiation is then:

$$f(x) - f(x-1)$$

It is understood that the independent variable x can take on only integer values, since its smallest value is 1 and any larger values must be integer multiples of 1. The dependent variable is often also integer valued, but may not be (see paper on ‘Uncompleted Operations’).

In this study only polynomial functions will be examined, so we start with finding the quantum differential equations of powers of x. If $f(x) = x$ then the backward difference equation yields just 1,

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hence the quantum derivative of x is 1. For $f(x) = x^2$ applying the backward difference formula yields the derivative 2x-1. In fact, we find the first several values of the derivatives of powers of x to be:

f(x)	derivative of f(x)
c	0 (constant function)
x^1	1
x^2	$2x-1$
x^3	$3x^2-3x+1$
x^4	$4x^3-6x^2+4x-1$
x^5	$5x^4-10x^3+10x^2-5x+1$
x^6	$6x^5-15x^4+20x^3-15x^2+6x-1$
x^7	$7x^6-21x^5+35x^4-35x^3+21x^2-7x+1$
x^8	$8x^7-28x^6+56x^5-70x^4+56x^3-28x^2+8x-1$
x^9	$9x^8-36x^7+84x^6-126x^5+126x^4-84x^3+36x^2-9x+1$
x^{10}	$10x^9-45x^8+120x^7-210x^6+252x^5-210x^4+120x^3-45x^2+10x-1$
x^{11}	$11x^{10}-55x^9+165x^8-330x^7+462x^6-462x^5+330x^4-165x^3+55x^2-11x+1$
x^{12}	$12x^{11}-66x^{10}+220x^9-495x^8+792x^7-924x^6+792x^5-495x^4+220x^3-66x^2+12x-1$
x^{13}	$13x^{12}-78x^{11}+286x^{10}-715x^9+1287x^8-1716x^7+1716x^6-1287x^5+715x^4-286x^3+78x^2-13x+1$

Table 1: Quantum Differentiation

Using the notation of ‘D’ for quantum differentiation (and later, ‘I’ for quantum integration) we also find that D is linear:

$$D(cx^n) = cD(x^n), \text{ c a constant coefficient, and}$$
$$D(x^n + x^m) = D(x^n) + D(x^m),$$

and other standard calculus identities hold as well.

Several characteristics of the above table of quantum differentiation are intriguing. The coefficients seem very close to those of Pascal’s triangle. Starting with Pascal’s triangle if we eliminate the leading 1 from each row (thus eliminating the first row altogether) and then negate every other term starting with the now second term of each row, we arrive at exactly the above table of coefficients. This seems rather remarkable, but is due to the backwards difference formula: it is simply a binary expansion of a difference less the leading term (*f(x) - f(x-1)*). If we sum the coefficients of any row of the above table we find they always sum to a value of 1; again, rather remarkable. And, if the coefficients of Table 1 are entered into a lower triangular square matrix the matrix is involutory, that is, it is its own inverse.

As opposed to continuous calculus which defines a slope at every point (at every real value of x for continuous differentiable functions) quantum calculus can only define the dependent variable change between two adjacent independent variable values. There are no points on a line representing slope since the values represent discrete quantities and there are no intermediate values between indivisible units of measure. Consequently we can *correctly* speak only of the change in *y* with unit change in *x*, not the slope of the line between two values of *x*. When we solve for the rate of change at ‘point’ $x = 3$, for example, using the above formulas, we obtain the change in *y* from *x=unit 2* to *x=unit 3*.

We now have two methods of finding quantum difference equations: the backwards difference formulation and the modified Pascal’s triangle formulation. With the quantum differentiation formulas in place we can move on to establishing the integration formulas for polynomials. Today there are

several methods of developing the integration formulas, some rather esoteric. Historically, Bernoulli, Faulhaber, and others, were solving the sum of powers of integers problems, which led to the Bernoulli numbers and a general, although rather complicated, solution. An example of the type of problem posed is: what is the sum of the first 5 integers each raised to the 2nd power? **This, of course, is also the quest for the quantum integral of x^2 from 1 to 5.** Thus if we know the formula for the sum of cubed integers we also know the formula for the quantum integral of x^2, and the reverse is true as well. A diagram of values, rates of change, area under values, and sum of powers of integer concepts for quantum calculus makes this plain:

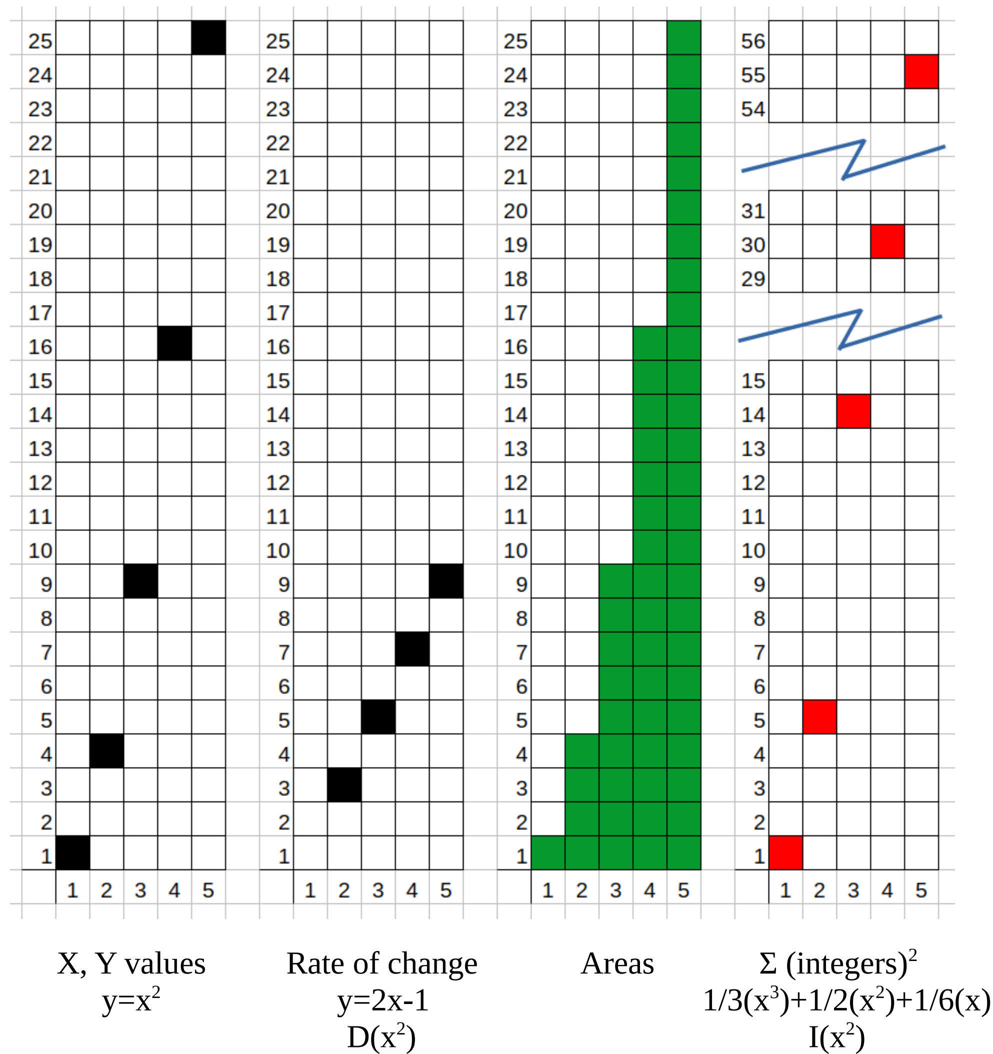

Table 2: Quantum Differentiation and Quantum Integration

It's tempting to draw lines between the squares of the first, second, and fourth plots but we cannot. A point on such a line would of necessity subdivide the smallest lengths labeled 1, 2, 3, 4, and 5. The Areas plot shows the equivalence of quantum area and sum of powers of integers. Notice that the coordinate system labels 'spaces' not 'lines' and that the axes start with 1, not 0. This is a quantum graph on quantum graph paper.

Before rediscovering the Bernoulli numbers we must digress. Several conventions have arisen around the Bernoulli numbers that must be explicitly stated and, perhaps, improved upon. From algebra we routinely express polynomials in descending order of degree, as in Table 1. This is pure convention (as we'll see when we impose a modified Pascal's *triangle* on a *square* matrix in at least eight ways). The actual Bernoulli numbers are sometimes put in table form omitting all zero values, sometimes retaining the zeros, sometimes the second number is written as a positive ½, and sometimes as a negative ½. When the expressions for the sums of integers to a certain power are written they sometimes include the zero coefficient terms, sometimes not, and are sometimes written in descending order of degree of term, sometimes not, changing how the terms are labeled (numbered). Further, the conventions of matrix manipulation are embedded in standardized calculation procedures. Taking the inverse of a matrix, for example, depends on row calculations giving rise to seemingly non-symmetric results, demonstrated later.

In this paper the following conventions apply unless otherwise explicitly stated:

- Polynomials will be expressed in descending degree of terms
- Polynomial terms are numbered from the left starting with 1, regardless of order of degrees
- The Bernoulli numbers are labeled (numbered) from zero
- The Bernoulli numbers include the zeros (every odd labeled number except B_1)
- The second Bernoulli number (B_1: ½) will always be positive
- Inverse quantum differentiation matrices do include zero terms
- Quantum integration formulas do not include terms with zero coefficient
- Quantum integration formulas entered into a matrix do include zero terms

There follows a direct brute force method for finding quantum integration formulas which builds upon the already discovered quantum differentiation formulas and is in keeping with discovering the underpinnings of a quantum calculus. Other, faster, methods are discussed later in this paper.

This technique for finding an integration formula for a power of the variable, x^n, relies on reiterative application of the previously determined differentiation formulas. Any candidate for the integral must conform to the composition of differentiation and integration functions yielding the original variable. Letting D(f(x)) be the quantum differentiation operator and I(f(x)) be the quantum integration operator we must ensure that:

$$D(I(f(x^n))) = f(x^n).$$

A simple example demonstrates the process, followed by a much longer example showing the intricacies giving rise to the zero coefficients of almost half the terms. First we find the integral of the variable x to the first power:

$$I(x^1) = ?.$$

$D(I(x^1))$ must equal x^1, so we try $\quad I(x^1) =?\ x^2.$

From the quantum differentiation formulas $\quad D(x^2) = 2x-1$, not x.

Next we try $\quad I(x^1) =?\ 1/2(x^2),$

but find $\quad D(1/2(x^2)) = 1/2D(x^2) = x-1/2$, still not x.

To eliminate the -1/2

we add a term to the integral $\quad I(x^1) =?\ 1/2x^2+1/2x$

and find $\quad D(\ 1/2x^2+1/2x) = x-1/2+1/2 = x$

so $\quad I(x^1) = 1/2x^2+1/2x.$

Next we find the integral of x^8, using the following table:

to find I(x^8)											1
								3	-3	1
						5	-10	10	-5	1
				7	-21	35	-35	21	-7	1
			8	-28	56	-70	56	-28	8	-1
		9	-36	84	-126	126	-84	36	-9	1
9	1/9	1	-4	9 1/3	-14	14	-9 1/3	4	-1	1/9
8	1/2		4	-14	28	-35	28	-14	4	- 1/2
			0	-4 2/3	14	-21	18 2/3	-10	3	- 7/18
7	2/3			4 2/3	-14	23 1/3	-23 1/3	14	-4 2/3	2/3
				0	0	2 1/3	-4 2/3	4	-1 2/3	5/18
5	- 7/15					-2 1/3	4 2/3	-4 2/3	2 1/3	- 7/15
						0	0	- 2/3	2/3	- 17/90
3	2/9							2/3	- 2/3	2/9
								0	0	1/30
1	- 1/30									- 1/30
		1	0	0	0	0	0	0	0	0

Table 3: Finding the Integral of x^8

Most of the arithmetic done in finding I(x^8) is done on the formula for D(x^9), so we enter that formula in the row between the bold lines. The aim is to add terms to the proposed integral that cancel out all the terms of the of the D(x^9) except the first term, leaving the first term with a coefficient of 1. The first term in the D(x^9) is degree 8 which is the goal of D(I(x^8)).

The coefficients of the derivative of x^9 are inserted in the cells between the bold lines. Then that row is multiplied by the inverse of the first coefficient with the result put in the next row down. To the left of that row is shown first the degree of that term *of the integral to be found*, and next the multiplier of that row, 9 and 1/9 respectively. We want to keep the first term, 1, representing $1x^8$,but wish to eliminate the second term, which is -4. To do that we enter in the row above the double bold line row the values of the derivative of x^8, then multiply that row by whatever it takes to eliminate the -4, in this case ½, and the values are entered into the next row and preceded by 8 indicating x^8 and ½ indicating the multiplier. This row is then added to the row above it with the results entered into the next row down. This process is continued until the last remaining unwanted term is eliminated. Notice that sometimes two terms are eliminated at the same time, which is why there are skipped degrees.

Finally, the answer is read from the two columns on the right, in this case:

$$(1/9)x^9+(1/2)x^8+(2/3)x^7-(7/15)x^5+(2/9)x^3-(1/30)x.$$

The integration table (Table 4, shown below) has several intriguing patterns, for some of which it is hard to quantify the underlying structure. The coefficient of the first term of an integration of any power function, x^n, is just 1/(n+1). The second coefficient is always ½. The third coefficient, starting with the integration of x^2, is a multiple of 1/12: n/12. The fourth term is always zero, as is every even term after that. Euler, Maclaurin, and others found general forms for the integration formulas, now called the Bernoulli polynomials. Their representation uses the B(0) (or B^{--}) convention where the second term's coefficient is made negative.

There are several additional fascinating properties of the table of discrete integration. First, despite the fractional coefficients, an input of an integer value of x always yields an integer value for I(f(x)). And like the derivative table, the sum of the coefficients of any indefinite integral (excluding a constant of integration) of x^n is always one. Finally, the coefficient of the x^1 term of the discrete integration of x^n

is Bernoulli number B_n. These calculations produce Bernoulli numbers consistent with the convention $B_0=1$, $B_1=1/2$, $B_2=1/6$, and $B_3=0$, yielding:

f(x)	indefinite integral of f(x)
c	1/1(x) (constant function)
x^1	$1/2(x^2)+1/2(x)$
x^2	$1/3(x^3)+1/2(x^2)+1/6(x)$
x^3	$1/4(x^4)+1/2(x^3)+1/4(x^2)$
x^4	$1/5(x^5)+1/2(x^4)+1/3(x^3)-1/30(x)$
x^5	$1/6(x^6)+1/2(x^5)+5/12(x^4)-1/12(x^2)$
x^6	$1/7(x^7)+1/2(x^6)+1/2(x^5)-1/6(x^3)+1/42(x)$
x^7	$1/8(x^8)+1/2(x^7)+7/12(x^6)-7/24(x^4)+1/12(x^2)$
x^8	$1/9(x^9)+1/2(x^8)+2/3(x^7)-7/15(x^5)+2/9(x^3)-1/30(x)$
x^9	$1/10(x^{10})+1/2(x^9)+3/4(x^8)-7/10(x^6)+1/2(x^4)-3/20(x^2)$
x^{10}	$1/11(x^{11})+1/2(x^{10})+5/6(x^9)-1(x^7)+1(x^5)-1/2(x^3)+5/66(x)$
x^{11}	$1/12(x^{12})+1/2(x^{11})+11/12(x^{10})-11/8(x^8)+11/6(x^6)-11/8(x^4)+5/12(x^2)$
x^{12}	$1/13(x^{13})+1/2(x^{12})+1(x^{11})-11/6(x^9)+22/7(x^7)-33/10(x^5)+5/3(x^3)-691/2730(x)$

Table 4: Quantum Integration

There are other ways to find the quantum integration formulas. Since they are similar to those of the 'sum of powers of integers' formulas the techniques for finding the sum of powers of integer formulas also finds the quantum integration formulas. There are generalized formulas for the sum of powers of integers that work for any power of the variable given a table of known Bernoulli numbers, and there are formulas for finding the Bernoulli numbers, as well. There are formulas that combine the two. These generalized formulas are somewhat cumbersome, involving falling factorials, factorials, the Bernoulli numbers, combinatorics, summations, and double summations of many terms.

Ignoring the 'sum of powers of integers' path altogether and looking only at the simply derived quantum differentiation formulas there is an absolutely fascinating way to find the quantum integration formulas, and so also the sum of powers of integers formulas as well as the Bernoulli numbers.

If the coefficients of Table 1: Quantum Differentiation are entered into a lower triangular matrix reversing the order of the terms, and then the inverse is taken we obtain the matrix shown here:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{12} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{42} & 0 & \frac{-1}{6} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{12} & 0 & \frac{-7}{24} & 0 & \frac{7}{12} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{30} & 0 & \frac{2}{9} & 0 & \frac{-7}{15} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{9} & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{20} & 0 & \frac{1}{2} & 0 & \frac{-7}{10} & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{10} & 0 & 0 & 0 \\
\frac{5}{66} & 0 & \frac{-1}{2} & 0 & 1 & 0 & -1 & 0 & \frac{5}{6} & \frac{1}{2} & \frac{1}{11} & 0 & 0 \\
0 & \frac{5}{12} & 0 & \frac{-11}{8} & 0 & \frac{11}{6} & 0 & \frac{-11}{8} & 0 & \frac{11}{12} & \frac{1}{2} & \frac{1}{12} & 0 \\
\frac{-691}{2730} & 0 & \frac{5}{3} & 0 & \frac{-33}{10} & 0 & \frac{22}{7} & 0 & \frac{-11}{6} & 0 & 1 & \frac{1}{2} & \frac{1}{13}
\end{pmatrix}$$

Table 5: Inverse of Quantum Differentiation Matrix

Notice two things: First, **each row is the quantum integral** for the corresponding power of x (row n is the formula for $I(x^{n-1})$), in order of ascending degree starting with degree 1.

And second: the **Bernoulli numbers are displayed** in the first column.

All of this with just the push of a calculator button to invert a matrix!

The differentiation formulas must be entered in reversed order (lowest degree term first). Investigating this requirement brings to light some additional interesting aspects of the quantum differentiation coefficient matrix.

Strangely, if the differentiation coefficients are entered highest degree first in a lower triangular matrix the matrix is involutory. That is, the matrix is its own inverse, the inverse yielding the differentiation matrix again, rather than the integration matrix.

Starting with that involutory matrix there are three things (at least) we can do to rearrange the entries. We can right or left justify the terms within the matrix, we can reverse the order of the terms, and we can enter the rows starting at the bottom of the matrix (upside down, the single term row at the bottom of the matrix). We can abbreviate these rearrangements right/left for justification, std/rev for standard (highest degree first) or reversed order, and nrm/inv for normal (single term formula in row 1) or inverted (single term formula in bottom row). This gives us (without using transposition) eight permutations. Due, apparently, to the matrix inversion algorithm there is no arrangement that yields a left/std/nrm matrix of integrals from a left/std/nrm matrix of differentials.

As noted, the inversion of the left/rev/nrm differential matrix yields the left/rev/nrm differential matrix. There are two such involutory arrangements: left/std/nrm and right/rev/inv.

There are two cyclic involutory arrangements, that is, the inverse of one yields a rearrangement of the entries the inverse of which brings back the first arrangement: right/rev/nrm to left/std/inv.

There are four arrangements whose inverse does yield the integral coefficients;

left/rev/nrm to left/rev/nrm, right/std/nrm to left/rev/inv,
left/rev/inv to right/std/nrm, and left/std/inv to left/std/inv.

Of these the first and last are most useful – the first generated Table 5.

Also of note, if the matrix of quantum integration coefficients are entered in reversed order (the order in the above table) that matrix is not invertible.

Historically, when a discrete form of differentiation was desired a very different and strange approach was used. The standard approach to a form of discrete (or finite) calculus involves finding a different form of the operation of exponentiation. In standard calculus the derivative of x^2 is simply $2x^1$, so a new type of exponent that behaves the same way was needed. That new exponent is a 'falling powers' type of number. (See the paper "Finite Calculus: A Tutorial for Solving Nasty Sums", David Gleich, Jan 17, 2005. Web site URL:
https://www.cs.purdue.edu/homes/dgleich/publications/Gleich%202005%20-%20finite20calculus.pdf)
A falling power is denoted by underlining the exponent: $x^{\underline{2}}$, for example. It is defined as:

$$x^{\underline{m}} = x(x-1)(x-2) \dots (x-(m-1)).$$

This strange looking construction does satisfy the desired outcome: $Dx^{\underline{m}} = mx^{\underline{m-1}}$. To actually use the integration theorems defined using these falling powers, one must be able to convert regular exponents to falling powers first, a rather taxing prospect. It's much easier to use quantum integration formulas directly.

In Summary:

This paper demonstrates a quantum form of calculus, both differentiation and integration, that yields correct answers for any process that depends upon discrete indivisible units. Processes that are quantum-like may find this form useful, and unlike classical calculus, discrete calculus has no ambiguities around infinitesimals, division by zero, or ignoring non-zero values.

The combinatorial formula for generating the quantum differential of the terms of a polynomial (not shown herein) suffices for physical calculations, however, the general formulas for the sum of powers of integers (hence quantum integration) are cumbersome. The inverse matrix method of finding the quantum integration formulas (hence the sum of powers of integers formulas) is very simple.

There is the interesting equivalence that arises from these efforts. Letting the integral symbol with ‘Q’ in place of the lower limit and ‘n’ as the upper limit indicate a quantum integration evaluated at n, we have:

$$\sum P(n) = \int_{\mathrm{Q}} P(x)$$

That is, the sum of a power of the integers polynomial evaluated at integer values from 1 to n is equal to the quantum integral of that polynomial evaluated at n. For example:

$$\sum_{n=1}^{t} (2n^3 + 5n^2 - n) = \int_{Q}^{t} (2x^3 + 5x^2 - x) \quad ,$$

where

$$\sum_{n=1}^{t} (2n^3 + 5n^2 - n) = 2 * \sum_{n=1}^{t} (n^3) + 5 * \sum_{n=1}^{t} (n^2) - \sum_{n=1}^{t} (n) \quad ,$$

a sum of sums of integers to powers, which yields from the quantum integration tables:

$$\frac{1}{2} t^4 + \frac{8}{3} t^3 + \frac{5}{2} t^2 + \frac{1}{3} t \quad .$$

Setting aside sum of powers of integers calculations and combinatorial factorial formulations, and looking only at quantum calculus formulas, we have the following straight forward set of conclusions:

1) the backwards difference formula yields the quantum differentiation formulas.
2) those formulas placed into an appropriate matrix form a modified Pascal’s triangle.
3) the inverse of that matrix yields the quantum integration formulas.
4) those formulas yield the Bernoulli numbers and sum of powers of integers formulas.

Since the backwards difference formula is a binomial expansion less the first term it is little wonder it yields a modified Pascal’s triangle. Since differentiation and integration are inverse functions it is small wonder that inverting the matrix might transform between the two. And since the sum of powers of integer equations are precisely the quantum integration equations it is no wonder the Bernoulli numbers are found in the inverse matrix of quantum differentiation.

It is known that the difference formulas (both forward and backward) yield modified Pascal's triangles, and it is known that a suitably modified Pascal's triangle in matrix form can be inverted to find both the sum of powers of integers formulas and Bernoulli's numbers. However, that the quantum differentiation and quantum integration formulas can be expressed as inverse matrices of each other is not generally realized, nor is the connection between quantum integration and Bernoulli's numbers.

While it is easier to remember via standard calculus that the integral of x^5 is $x^6/6$, then it is to remember the quantum integral of x^5 is $x^6/6+x^5/2+5x^4/12-x^2/12$, the later does actually have considerable merit. In this example, as with any of the quantum derivative and integration formulas, the *standard form is an approximation of the quantum form* and the ratio of the two becomes asymptotically close to 1 with relatively large values of x. Since almost every thing in physics deals with massively large numbers of 'smallest units' of things this approximation works fine. But when dealing with very small numbers of things, say at the quantum level, physics seems to be in trouble. Perhaps the old approximations can no longer be applied.

Additional Thoughts:

This exploration of quantum calculus is part of a larger effort to re-cast mathematics into a positive integer only version that is fully as capable in describing our physical world as current mathematics, and does so in a more understandable manner, possibly explaining some nagging quandaries of physics. Surprisingly, calculus is easier to 'quantize' then most of arithmetic.

This look into a different way to define discrete calculus demonstrates how most of our standard math has been constructed. There are two ways to modify a math operation when it doesn't meet all of our needs. One is to change the operation, the other is to change the set of elements it operates on. By far the most common approach is to modify the set of elements, that is add new numbers to those already available for use. In the above *discrete calculus* example a falling power was created. In *quantum calculus* the definition of the operation of taking a derivative was modified: the *h* of standard limit theory was replaced with a 1 and limits were not needed.

As an example of this dichotomy from arithmetic, taking much historical license: when subtraction was first invented everything was fine; 4 sheep take away 3 sheep left 1 sheep. Then some rabble-rouser asked what about 4 subtract 5? We promptly invented zero and the negative numbers. What we could have done, instead of enlarging the set of numbers to include numbers that do not apply to the real world, was change the operation. There are two ways that can be accomplished, one is to allow only smaller numbers to be subtracted from larger numbers, by the new definition of subtraction. The other way is to redefine subtraction so that it results in the difference, if any, of two numbers. This would even be commutative: a-b = b-a, which in our standard math would be |a-b| (the absolute value of the difference). An additional operation would then be necessary, to identify the larger of two numbers. Both ways have serious difficulties that must be dealt with before being useful.

Modifying the operation requires the difficult task of breaking an entrenched paradigm. Modifying the number set is much easier, but it brings into mathematics much that cannot apply to the real world. Subtraction has the indicated problems, division is much worse to envision, but it would finally demonstrate the difference between a rational number (uncompleted operation) and a ratio (comparison of two numbers). Creating new numbers (zero, negative, rational, irrational, infinite, imaginary, etc.) is really just a way of postponing an operation, hoping that it will somehow cancel out of the equation before the result is reached. These postponements I call uncompleted operations.

As a working hypothesis, in the real world ‘things’ either exist or they don’t. If they exist they are in some quantity of a least unit. That quantity is always a positive integer value. If a physics calculation yields a negative result it is always interpreted as being ‘in the opposite direction’ or ‘the opposite charge’, etc. In finance negative money is ‘debt’. Always before applying a negative number to the real world it must be interpreted as some positive amount with a different meaning.

Another hypothesis driving this positive integer mathematics construction is that things that exist are finite in number and their smallest unit is indivisible. An example from physics is that of the electron. It is the smallest unit of charge and it cannot be divided. Any amount of ‘electricity’ must be an integer multiple of the charge of a single electron.