The Role of Uncompleted Operations in Number Systems

Robert Duncan

Introduction

Thesis:

Most mathematics text books, particularly abstract algebra books, find little need to derive or explain the ‘natural’ numbers. The operation of addition, and to a lesser extent subtraction, can also be taken as needing little explanation or verification. Counting with digits (Latin *digitus*. finger) sufficed for tens of thousands of years.

However, the desire to solve the preimage problem, that is, the drive to create inverse functions, forced the creation of extensions to the natural numbers. More abstract and complicated arithmetic beyond just addition and subtraction created iterative operations such as multiplication and exponentiation, which drove the need of yet more inverse function related extensions to the number system. Iteration requires the indexing or counting of operations, a use of number still not widely recognized as such. These extensions to the natural numbers include all forms of numbers other then the positive integers. Put another way, the natural numbers form the basis for all the extensions of our number system, and those extensions are the ‘unnatural numbers’.

It is the thesis of this work that all of these extensions are of the form of uncompleted operations, and that any equation or function created to explain or quantify a property of the universe, to be rigorously correct (although in the macro world needlessly unmanageable), must be obtained from data that is drawn only from the positive integers, and whose result, to be applicable to the universe, must be expressed in positive integers only.

Properties of discrete finite objects have as their least amount the value 1 at their inherent smallest scale. Further, our current mathematics allows uncompleted operations in support of inverse functions, which, when adopted by the physical sciences leads to questions about zero, infinity, and space and time. Mathematicians may have to relinquish their grip on preimage functions and inverses, and recognize that to encompass the real world some operations may not be undone.

This paper attempts to both explain and support these propositions.

Consequences:

If successful, this new paradigm will force a re-casting of most of mathematics. It does not replace, correct, or substitute for existing mathematics, instead it allows new insight into the use of such tools as the real number line, real numbers, complex numbers, and calculus, to name just the most obviously affected areas.

Applications:

A possible view of the universe is that it and all its properties are at some level both finite and discrete. Our current very extended mathematics is both infinite and continuous. This paper hopes to prompt a search for a mathematics better suited to the finite and discrete systems which seem indicated by recent research and theory, so that mathematics can continue to support the physical sciences in whatever direction they may lead.

Conventions:

There are several conventions and definitions used throughout this paper that must be acknowledged at the start. First, ‘natural numbers’, while not consistently defined within the mathematics community, are defined herein as being the positive integers – specifically excluding zero as neither a number or natural. Another name for these numbers is ‘counting numbers’. While it is true that programmers generally start counts with zero, and the sigma, pi, and integration notations can index from zero, rarely does anyone else count other than by starting with one and adding one for each instance of the things counted. Also, counting past finite numbers is never actually done, for obvious reasons.

Next, an ‘operation’ is a mathematically defined function or relation converting one or more numbers into one number; the normal unary and binary operations involving one number and an operation (5! is unary factorial) or two numbers and an operation (3+4 is binary addition) are examples. Some operations that are commonly held to be binary but, as explained later, are not, include multiplication, division, exponentiation, etc. Actually the factorial expressed as n! is not unary but n-ary, as will become apparent later. Other operations or processes exist, such as cancellation, that can be examined and defined as the need arises.

A key concept of this paper, that of ‘uncompleted operation’, is an indicated operation that can not be completed within the scope of the positive integers, forcing an extension to the natural number system. That extension is almost always formed by defining the two (or more) numbers involved along with the associated operation symbol as a *single number* of the extended number system. The next section of examples will make this very important concept clear.

We will run into many examples of symbols composed of multiple symbols in this paper. The terminology used herein is: ‘simple symbol’ – a single symbol occupying, in general, a single font type space; and ‘compound symbol’ – a single symbol made up of two or more simple symbols occupying, in general, two or more font type spaces. Examples of simple symbols include: 3, +, and /. Compound symbols include: 3/8, 4-2*i*, and -3.

Technically, numbers that are written as Hindu-Arabic symbols generally fall into the category of compound symbols composed of the simple symbols of the ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, and 9). Examples such as 25, 17, and 101 are compound while 3, 8, and 0 are technically simple symbols. Number systems can have different bases and different properties. In general, this paper will consider numbers as single entities divorced from the system used to write them. Any number without other symbols is a simple symbol. *Therefore, the examples of 25, 17, and 101 are defined as simple symbols*, while -25, 3/8, and 3*i* are compound.

The confusion arises because most number systems express numbers as hidden multiplicative operations. The symbol ‘25’ represents (5 * 1) + (10 * 2), a very compound symbol, as opposed to the number ‘25’ as a simple (single) entity. Here we consider all positive integers as being simple symbols (numbers) or compound symbols (expressions), as the context dictates.

The symbol ‘*’ is used to indicate multiplication, as in 3 * 5 = 15. Occasionally, when no ambiguity exists, the product of two variables is shown by setting them together: a * b = ab.

Finally, the discrete, finite, positive integer math proposed by this paper is abbreviated PIM, Positive Integer Math.

Section 1: Examples of Uncompleted Operations

Rational Numbers:

A rational number is defined as the quotient of two integers. In some elementary texts it is defined as the ratio of two integers. The second definition is actually incorrect, as ratios express more information than do fractions, and ratios are best expressed not as quotients but as two numbers separated by a colon, as in 3:2. A flipped coin whose history is one head in two flips (a ratio of 1:2) is very much less likely to be 'fair' than a coin whose history is 500 heads in 1000 flips (a ratio of 500:1000), yet both expressed as the quotient of two integers (fractions) convey only the information contained in '½'.

Most rational numbers can not be reduced to an integer, that is, when viewed as a fraction the numerator is not an integer multiple of the denominator. That being the case, most rational numbers must be indicated as what appears to be a fraction, but is not. The *fraction* 3/8 can be expressed as 0.375. A change of scale could make this number 375. The *rational number* 3/8 is a single compound symbol: '3/8'. It can not be represented in any other form. The symbol '/' within the symbol 3/8 is not a division sign, it is part of the single symbol '3/8'. '3/8' is not an integer, it is an extension to the system of integers that allows for divisions that do not result in an integer. It allows for the apparent calculation of the value of x in the equation 3 = 8 * x.

The function f(x) = 8 * x, x an integer, is a straight forward function, but its inverse, g(x) = x/8 is not, and may not have an integer solution. This desire for an inverse function (the preimage) drives the need to extend the integers to include a new form of number, the rational numbers.

The fact that 3/8, as a single compound symbol which includes a division symbol that is not a division operation does not keep us from pretending that it *is* a division operator. If we multiply 3/8 by, say, 24, we get 9, by first *multiplying the 3 times 24* yielding 72/8, *and then dividing 72 by 8*. Worse, if we divide 3/8 by, say, 3/7, we *invert the divisor* obtaining 7/3, and then multiply (not cancel) in a special way. If 3/7 is really a single symbol how can it be 'inverted'? Each rational number has a defined multiplicative inverse, the symbol for which appears to be the 'inverted' original symbol. Note that the special way we add and multiply rational numbers with themselves and with natural numbers comprises a special algebra, with rules quite different from adding and multiplying natural numbers alone.

Rational numbers are uncompleted divisions of the natural numbers.

Complex Numbers:

Complex numbers are composed of several uncompleted operations and a misunderstanding. The standard complex number is defined as: 'a + b*i*, with a and b included in the real numbers, and *i* the 'square root of -1'. The symbol *i*, the square root of -1, is an imaginary number. Right away we see an uncompleted operation: addition of a real with an imaginary is not a defined operation. Also, an uncompleted multiplication of a real with an imaginary is also indicated, as well as an uncompleted square root of an uncompleted subtraction of an integer from zero. Yet, the symbol a+b*i* is supposed to be a single symbol representing a single number. An example: 2 + 3*i* is really 2 + 3 * √(0 – 1).

Although defined as a single compound symbol, a+bi is treated as if it were indeed addition and multiplication as indicated: the sum of a+b*i* and c+d*i* is the single compound symbol e+f*i*, where e = a+c and f = b+d, and the product, formed by distributing multiplication over addition and using special algebra for *i*, is the single compound symbol e+f*i*, where e = ac-bd, and f = ad+bc. One cannot help but notice that the single compound symbol a+b*i* is treated in calculations as a series of uncompleted operations on several single simple symbols and not as a single simple symbol at all.

The misunderstanding referenced above in dealing with complex numbers lies with how many imaginary numbers have been defined. Most mathematicians routinely assume that there as many imaginary numbers as there are real numbers, after all, one simply affixes (multiplies by) an *i* to any real number and the corresponding imaginary is created. In fact, there is only one imaginary number, the square root of negative one, and an infinite number of uncompleted multiplication operations by *i* on the reals.

Complex numbers are actually composed of several uncompleted operations on the natural numbers.

Negative Integers:

Most abstract algebra texts that develop the full set of integers from the natural numbers do so in an appendix or some other out of the way location. The arguments differ slightly depending on the author's definition of natural numbers (with or without zero), but follow the schema briefly presented here.

Create a new set of numbers, each one represented by a compound symbol of two natural numbers within a set of parentheses: (a, b), where a and b are natural numbers. Now define an equivalence relation on these numbers as (a, b) ≡ (c, d) iff a+d = b+c. Right away we see the single symbol (a, b) is being treated as some kind of uncompleted operation.

With this equivalence we notice many elements in each equivalence class and can select representative elements for each equivalence class. One such selection is: (a, b) is a class representative iff a or b or both are the natural number 1. Another common selection is: (a, b) is a class representative iff a or b or both are zero, if zero was included in the natural numbers to begin with.

Addition is defined as follows: (a, b) + (c, d) = (a+c, b+d) and subtraction, the inverse function that causes problems with the natural numbers, is defined: (a, b) – (c, d) = (a+d, b+c).

Notice that for equivalence, addition, and subtraction the calculations involve only addition and the natural numbers (positive or non-negative depending on the inclusion of zero), so the bother of having to extend the positive integers into the negatives to allow subtraction of a larger number from a smaller number is resolved, and any subtraction yields a result within the defined number system. But much is hidden, as a actual example will demonstrate.

To ensure that this new set of numbers is useful it must be ordered. Briefly, to allow for this ordering a 'unit' is defined, in this case the class represented by (2,1). Then the 'successor' to every number is defined as the representative of the equivalence class of the sum of that number plus the unit. This orders the set of new numbers.

In analyzing these numbers we find that some numbers when added to a given number produce a result larger in a ordered chain of successors than the given number while some other numbers produce a smaller number in a chain of predecessors. This allows us to create a shorthand symbology: the ‘negative integers’. A negative integer added to any other integer produces a result that is less than the other integer.

The mapping of equivalence class representatives of these new numbers to the integers is: (3, 1) → 2; (2, 1) → 1; (1, 1) → 0; (1, 2) → -1; and (1, 3) → -2, to show just a few numbers around zero. Obviously, the single symbol (a, b) is actually the uncompleted operation a – b. It is even more apparent if we had used class representatives that have one or both internal numbers zero. Then (2, 0) → 2; (0, 0) → 0; and for the negatives (0, 2) → -2. Here we see that the uncompleted operation of subtraction of the second natural number from the first can be represented in a shorter way, which is this: (0, 2) → 0 – 2 = -2 and we’ve arrived at the negative integers.

To be sure, -2 does not represent a number, but an uncompleted operation of the subtraction of natural numbers.

There is a similar confusion about how many negative numbers exist as there is about how many imaginary numbers exist, and they have the same resolution: one. *Just negative one (-1) needs be defined, all the rest of what we consider negative numbers are just positive numbers times -1, which is, of course, an uncompleted operation.*

Real Numbers:

A real number, rational or irrational, if it has a decimal part is an uncompleted operation. For example, the number 1.5 is 1 + ½ which is an uncompleted addition and an uncompleted division. The number 1.5 can be expressed as 3/2 which is still an uncompleted division. An approximation to negative *pi*, for example, is obviously an approximation, but is also an uncompleted subtraction (0 – *pi*) and an uncompleted division (the decimal part). The uncompleted division arises from the definition of *pi*, the circumference of a circle divided by its diameter. Of course, the lack of completeness is made worse by the fact that the two numbers are incommensurate.

In short, every number aside from the positive integers is an uncompleted operation. And, as we’ll see later, even the symbols we use for the positive integer numbers are uncompleted operations!

Section 2: Finite Positive Integer Processes and Algorithms for Operations

Every Calculation is a Positive Integer Process:

Every mathematical operation has a defined process or algorithm for its resolution. To demonstrate we can use the addition of two complex numbers: (123.45 + 76.2*i)* and (34.001 + 10*i*). Without belaboring the actual mechanical steps:

First line up the real part of each number one over of the other so that the radix points are in the same column. Then add zeros as necessary to either end of each number so that there are the same number of digits in every column of digits. Then starting on the right, *add the positive integers* keeping track of carries as needed. Now do the same for the imaginary part of both numbers.

What if one of the numbers had been negative? Well we can 'change the sign and subtract', so that we are subtracting *positive integers*.

Examination of any operation reveals that the process for resolution involves working only with positive integers and addition or subtraction. Division, multiplication, exponentiation, even integration and differentiation have algorithms for their solutions that involve only positive integer arithmetic.

The extensions of numbers and operations beyond the natural numbers and addition serve only to facilitate human formulation and understanding of relationships that exist in the real world. Of themselves, the extensions to natural numbers add no substance to mathematics. They can, however, create false or at least unsubstantiated precepts of the real world. As an example, consider the obvious property of rational numbers that there is no smallest rational – the number line can be subdivided indefinitely meaning that between *any* two numbers (rational or irrational) there is always a rational number bigger than the smallest of the two and smaller than the largest. This ingrained paradigm causes us to think that space, time, and, before quantum physics, matter and energy, are also infinitely divisible; an everywhere dense continuum of space, time, matter, and energy. This is not the case for matter and energy and indications are that it may not be the case for space and time.

Physical sciences that make use of numbers other than the natural numbers are actually using those numbers in place of a different algebra for different quantities. Stand up and try moving east +3 steps then moving *east -4 steps* and you find it is really moving east 3 steps then moving *west 4 steps*. Adding 3 steps east to -4 steps east is really adding 3 steps east to 4 steps west. The revised algebra for addition of east and west movements is this: adding two of the same direction, just add the values and affix the common label; adding two of different directions, find the difference (the absolute value of a subtraction of one from the other) and affix the label of the larger of the two. Similar arguments can be formed for calculation with imaginary numbers, rational numbers, etc.: all can be restructured to operate with only positive numbers and a revised algebra designed to work specifically with the properties or objects being manipulated.

Computer Calculations:

This is the age of the digital computer. 'Digital' means discrete as opposed to analog which is continuous (to within the limits of the physical components). These wonderful devices can calculate orbital docking maneuvers and precision laser machining at speeds that seem like magic to those of us that learned on 300 baud audio modems and 'massive' 64K memory banks. The point here is that all this magic is done within the computer processing chips using what amounts to positive integer arithmetic. This fact alone points to the validity of the first part of this section.

Computers do calculations using registers of a fixed length. The first registers were only 8 bits ‘wide’, meaning the registers could store only 8 binary digits. Now 64 bits is common in PCs. Numbers are stored, on hard drives for example, in various lengths depending on the number type, but always of fixed length (finite) for each defined type of number (for any particular programming language’s number types, e.g. double, integer, etc.). Examining the registers where the numbers are handled for calculations we see that only binary integers can be manipulated. Often the leftmost bit of a register is the sign bit, leaving one less bit for the number. The sign bit could be considered to be a separate one bit register which means to manipulate a signed number takes two registers. Two registers (or four if the two numbers are signed) can be used to manipulate a rational number, and multiple registers can be used to manipulate a *finite approximation* of an irrational number. Multiple registers are also used for complex numbers, quaternions, vectors, tensors, etc. (This description of register manipulation is somewhat over-simplified; my apologies to fellow programmers.)

Within a computer it is apparent that only finite positive integers can be stored in a single register and thus only positive integers are used in calculations. *Using more than one register to hold a ‘single’ number is the equivalent of using uncompleted operations in mathematical notation.*

Inverse Functions and Numerical Inverses:

As mentioned in the Thesis section of the Introduction to this paper, being able to ‘undo’ an operation mathematically has driven the introduction of all types of numbers beyond the positive integers. Group theory defines rings, fields, and other algebraic structures based upon, among other things, whether or not a set of objects (numbers, generally) includes additive and multiplicative inverses, which implies the existence of those identities as well. Function theory defines surjective, injective, and bijective functions (old version: onto, one-to-one, and both) based on whether an unambiguous inverse function exists. When all of these considerations are at their ‘best’, mathematics, particularly algebra, ‘works’ without any problems: any operation can be applied to any expression and a consistent, unambiguous, well defined answer will be obtained. (Well, almost without problems – zero still manages to get in the way.)

The advent of calculators has driven home the idea of four basic functions of mathematics – addition, subtraction, multiplication, and division: the ‘four function’ simple calculator. As calculators became cheaper, faster, and more prevalent, more functions were included, up to programmability of almost anything. But we still think in terms of four basic functions of algebra. Group theory generally deals with one or two operations that can be defined however one wishes. Two operations are necessary if some sort of distributive law is desired in the algebra.

Addition is indeed a basic operation but multiplication is not (nor is subtraction or division). There are several kinds of multiplication, they just are not recognized as being as ‘basic’ as the multiplication of the four functions. Three examples of common types of multiplication beyond simple multiplication are the factorial function, the pi notation (see Section 4), and exponentiation. Much less known are various types of division, the most relevant here is a particular flavor of partitioning.

Oversimplifying a bit, partitioning a set of objects means dividing it up into parts each of which has at least one element from the original set. Working with integers as an example, 6 can be partitioned without regard to order into:

6, or
5, 1, or
4, 1, 1, or
2, 3, 1, or
1, 1, 1, 1, 1, 1, or
etc. Think of it as dividing 6 marbles into from 1 to 6 containers, leaving none empty.

Why is partitioning relevant to division? It is a generalization of common division. *Common division is not closed under positive integers but partitioning is closed.* Special constraints can be imposed on partitioning such as specifying the number of partitions or requiring that the parts must be as equal in size as possible. A further condition may be desired: some parts may be empty. It is then *integer division*, closed under positive integers. Call the operation '*indivision*'.

But there are three major differences between standard division and indivision. First, of course, is that while standard division is not closed for the positive integers, indivision is closed. Second, standard multiplication is the inverse function for standard division but not for indivision because: third, *standard division returns an unambiguous result, but indivision may not*.

We can define a function, id(n, p) = q, d_1, d_2. Given an input of the number of objects n (an integer), and the desired number of parts p (also an integer) the function returns three integers: q – the first quotient, d1 – the number of parts with that quotient, and d2 – *the number of parts with q+1 as the quotient*. Some examples:

id(6, 3) = 2, 3, 0   3 parts with 2 each and none with 3
id(6, 4) = 1, 2, 2   2 parts with 1 each and 2 parts with 2 each
id(6, 7) = 0, 1, 6   1 part with 0 and 6 parts with 1 each (assuming zero is allowed)

In general, the indivision function will return two quotients with a difference between them of one. A simpler function could be defined, ids(n, p) = q, it being understood that there may be another solution, q+1, as well. A simple test would indicate if there were another result: if p*ids(n, p) = n then ids(n, p) has only one result.

As an example of standard inverse functionality:
6/4 = 1.5, and 1.5*4 = 6, but
ids(6, 4) = 1 and with testing 4*1 ≠ 6, so 2 is also a result,
however, 1*4 ≠ 6 and 2*4≠ 6,
therefore *multiplication is not an inverse function of ids()*.

Restraining standard multiplication to the positive integers, it turns out that *id() is, however, an inverse function of multiplication*:

2*3 = 6 and
id(6, 3) = 2, 3, 0
since integer multiplication results in an integer multiple, of course!

There are many other divisions that can be defined using partitioning with constraints. Instead of requiring all parts to have as equal as possible number of elements we could ask that the number of elements in all the parts taken together follow a specified distribution, giving, for example a mean, sigma, and standard deviation. We could require that the parts collectively only need to average a certain number of elements in each part.

Given indivision as an operation, does an inverse operation exist, inmultiplication? Yes, but the point is made - an integer math can be constructed.

We've seen various types of multiplication and division, what about subtraction? Non-standard subtraction exists as well, and can be found in some programming languages. We can define such a subtraction function like this: When two positive integer numbers are subtracted in a non-standard way one of three possible results occur. If the first number is greater than the second number (5-3) a positive integer results (2). If the two numbers are equal a 'no longer exists' results, commonly called zero. If the second number is larger the operation is undefined. We typically define zero and negative integers as numbers and just continue on our way, but reality is not beholden to mathematics. If you only have two goats and sell two goats you no longer have any quantity of goats. Worse, if you have only two goats you sell three goats you occupy either a different universe or a jail cell.

Addition seems to be immune to the requirements of modification to describe a finite and discrete universe, but there is at least one hidden condition on addition never considered. If, in all the universe, there are only 6 widgets and you have 5 of them, you cannot add 2 to your collection. In other words, in a finite universe any classification of objects will have a maximum count. You cannot add objects beyond that maximum.

An algebra without inverse functions may describe the universe better than the current model. Physics claims that there is no reason experiments could not run backwards, in time or space. But that assessment may be based more on our mathematics that our reality.

Section 3: Language Shapes and Obscures Mathematics

Our Language Shapes Our Mathematics:

To an English linguist the two sentences, 'I have no money.' and 'I do not have any money.' are different grammatically but have the same meaning. To the common man in the street they are completely interchangeable, and he may not even hear the difference. To a discerning mathematician they are worlds apart in meaning and in philosophy.

Rephrasing the two sentences makes the difference a bit more apparent: 'I posses an amount of money equal to zero' and 'I do not posses any amount of money'. Clearly the first sentence speaks of possessing something and the second sentence denies any such possession. Yet both express the same condition of being financially disadvantaged. *Indeed, the second sentence would be nonsensical if zero were considered an amount: 'I do not posses any amount of money, including zero amount.'*.

Which brings us to the philosophical question, 'can one posses some thing in the amount of zero?' This blurring of meaning of the concept of zero happened as far back as when, in the early place value number systems, a dot indicating an empty column became a small circle then a larger circle than an actual digit of the number system. (See section 5.) This paper concludes that one cannot posses zero amount of any 'thing'. Zero is not a number or an amount, it can be viewed mathematically as the uncompleted operation of x – x, with x being a natural number. Zero can also be viewed logically as a statement of non-existence: '*I have zero money' means the state of my having money is non-existent.*

Difference Between 'How Much' and 'How many':

In English the question of 'how many' generally refers to a number of discrete objects that can be counted. In contrast, 'how much' refers to an amount of a continuous measure of some substance that can be divided into unlimited sizes. Grammar texts commonly define which expression to use based on whether or not the object of the expression is countable or not. Thus we have the expressions 'how much milk' and 'how many donuts'. Interestingly, many language authorities use money as an uncountable substance; 'how much money', 'how much profit', etc. If we were buying penny candy (is such a thing still available?) we would *not* ask 'how much pennies'. Since money is discrete and countable we should be saying 'how many money' and 'how many profits', as we do when asking 'how many dollars'.

The point here is that a native spoken language may not be a valid starting point for designing a mathematics, precisely because it is so completely internalized as to be unquestioned. There are many inconsistencies and much (many?) illogic in most languages which can form an unconscious and incorrect set of assumptions detrimental to mathematical rigor. Because we can grammatically say 'I have zero elephants in my pocket' does not mean that we should be able to translate that into a mathematically correct expression. We can, after all, *say* 'one plus one is three'.

The Many Problems with Negation:

An interesting exercise when driving a freeway is to try to change lanes without running over any of the little reflectors set in the pavement to delineate the lanes. This can keep the driver awake and ensure his passengers will be awake as well. There are two methods of attempting this: try to not hit the reflectors, and try to cross between the reflectors. In the first case one focuses on the reflectors and in the second case on the white lines between the reflectors. Many psychological studies have been done that indicate the mind does not process the word ‘not’, or better put, the mind operates on positive logic, and not on any form of negation. The result of this mental bias is that the first method of ‘not hitting the reflectors’ generally fails, while the second method, ‘hit the white lines’, generally succeeds. This is an example of incorporating negative logic into language that when used to ‘talk to oneself’ causes problems.

Mathematical examples are the negative integers and their extensions, which are the negation of the positive integers, defined mathematically as the additive inverses. Adding a number and its inverse results in zero. Adding a number and its negation is actually subtracting the number from itself. Hence negation and subtraction, two different operations, are so intertwined as to use the same symbol. Because we conceptualize negative quantities but can not mentally internalize them our minds grasp negative quantities as a different kind of positive quantity – that is, we mentally make negative quantities (and zero) ‘real’. They are not real and do not exist in the real world.

Examples of hidden negative logic abound in political statements, propaganda, and spin. This paper is not the forum for such arguments or even examples of them. With little effort one can find negative statements whose positive recasting makes no sense, and positive assertions whose commonly held negation does not have the original positive assertion as its complement or negation.

Section 4: Reiterative Operations and Index numbers

Multiplication, Exponentiation, and Index Numbers:

Addition of integers by counting is probably the most basic of all mathematics. For example, 3 stones plus 4 stones can be counted into a pile: one, two, three, and then one, two, three, four. The pile can now be counted: one, two, … , seven. So 3 stones plus 4 stones is 7 stones. So far so good. But now let's try basic multiplication. Let's count out 3 stones 4 times into a pile. This is a very different operation. When adding we added stones to stones and got stones. The set of natural numbers was applied to stones giving a set of stones closed under addition. The 3 and the 4 were both natural numbers *of stones*. However, when we examine multiplication we see that the 3 and the 4 are from two different types of numbers. The 3 is from the set of natural stones, and the 4 is from the set of natural numbers called 'pure' numbers; there is no label attached to the pure number. If they were both from the same set we would arrive at different answers: both from integers yields 12, an integer; both from natural numbers of stones yields 12 square stones, which has no meaning in the real world (1 stone is not a linear measure, so 12 square stones is not a measure of area).

The number four in this example is part of a mathematical construct, an algebraic operation. It is an index, a count, a replication. It is iterative addition. In this example 4 means 'add 3 stones to a pile 4 times' as an operation.

Because the 'new math' taught in high schools in the 1950's and 60's stressed such concepts as associative, commutative, and distributive properties of numbers many of us learned that 3 * 4 = 4 * 3 = 12. Previous to new math we learned that a multiplication operation had names for each part: a multiplicand, a multiplier, and a product (as did division, addition, and subtraction). A common definition of multiplicand taken from www.mathsisfun.com (a generally good site for youngsters) is: 'Multiplicand The number that gets multiplied.' But then, a la new math, they add, 'But because you can multiply the two numbers in any order, it is better to use the word "factor".' There is nothing wrong with teaching properties of numbers, and concepts such as the commutative property are very useful ideas applied to many areas of math. But it doesn't apply to many of the examples used in early school and shouldn't be taught there to the exclusion of the the older way of teaching operations.

Most definitions indicate that in the expression 4 * 3 the 3 is the multiplicand, the number that gets multiplied, and the 4 is the multiplier. Three stones taken four times would be written as 4 * 3 = 12, or better, 4 * 3 stones = 12 stones. Commuting the 'factors' yields the nonsense of multiplying the number 4 by 3 stones. *Multiplication in the world of objects is not commutative.*

The multiplier must be a 'pure' number, meaning not having a label, unit, or dimension. Coupled with an operation the pure number can be viewed as a part of the operation, a unary operation as opposed to a binary operation. Viewed this way, rather then the usual basic operations and their inverses (addition, subtraction; multiplication, division; and exp, log) we have an unlimited number of operations as, for example, every real pure number coupled with multiplication becomes a distinct unary operation. For example, we have the unary operator '3*'. Then 3* 2 stones = 6 stones, 3* 3 books = 9 books, 3* 21 coins = 63 coins, etc. We also have the unary operators '5*', and '341*', etc. We do not have unary operators with labels. That is, we don't have '3 stones*', or '2 dollars*'. It doesn't make sense to multiply by an object. Divisors and other operators that make use of a pure number can also be viewed as unary operators.

The Role of Uncompleted Operations in Number Systems
Robert Duncan

A close examination of what appears to be multiplying two labeled numbers demonstrates both what a rate (ratio) is and how it should be handled arithmetically. Suppose you earn $20 an hour and work for 3 hours. You expect to be paid $60: $20 * 3 hours = $60. But the real equation is ($20/1 hr) * 3 hr which is the uncompleted operation of a rate, $20 per 1 hour, times a labeled value, 3 hr. The hr labels cancel leaving $20 * 3 = $60. Any apparent product of labeled quantities must actually be a rate times an appropriately labeled value. One would not see 'miles per hour' times 'dollars per hour', or 'miles per hour' times 'stones'. Some measures appear to be multi-labeled: board-feet and acre-feet are two similar examples. They express volume: lumber equivalent to a board 12 inches wide by 1 inch thick by some length in feet, and irrigation of fields of so many acres to a depth of one foot. They are both better expressed mathematically in cubic feet, but that is not as graspable to a person used to fields measured in acres or 1 inch thick planks of wood.

Physicists have found it useful to define a limited number of 'basic' properties and use those properties in combinations to define all other properties of matter (and space-time). The standard system is called the MKS system which stands for meters, kilograms, and seconds. Additional basic properties must be added to these three, such as a unit for electrical charge, but for our purposes here we consider only the MKS system. A simple example from physics is that distance equals rate times time as given by the equation d = r * t. Distance is measured in meters and time in seconds. The rate is then measured as distance divided by time. In this example given a rate and a time we can multiply them together to find the distance. Here we are multiplying a rate with dimension M/S times a time with dimension S yielding a distance with dimension M. In this case rate times time seems to equal the time times rate: they seem to be 'factors'. But notice neither factor is a pure number. Also very importantly note: *we don't divide meters by seconds, the label seconds 'cancel out' of the equation*. M/S is an example of an uncompleted operation, carried along in hopes that it will 'go away' before the final answer is found, which fortunately it does. It cancels by design since the label M/S is derived from the equation r = d/t giving the rate a label of M/S. The actual final equation is a number (unlabeled) times meters (a labeled quantity); they are no longer factors but are multiplier and multiplicand.

The concept of cancellation of units (labels) is so compelling that constants are created with all kinds of strange labels so equations will 'cancel out' giving a real world result. The gravitational constant 'G' is one such example. From Newton we know that $F = GMm/r^2$, or the gravitational force, F, on a mass, m, by a mass, M, is equal to the gravitational constant, G, times the product of the masses divided by the square of the distance, r, between them. Solving for G we find the constant has units (labels) of a strange mix: newton*meter^2 / kilogram^2. A newton (N) is a measure of force with units of its own: kilogram*meter/second^2. Substituting we find that G has units in the MKS system of bizarre composition: meters^3/(kilograms*seconds^2). With those units and the appropriate pure number multiplier (6.67… x 10^{-11}) the equation is balanced, that is, when solved for any unknown quantity the units cancel appropriately. For example one can solve for how far apart two masses are with a given force and given masses and the result will have units of meters.

Meters^3 / kilograms is the specific volume of a substance (inverse density): how much 'space' 1 kilogram occupies. Interestingly, if this value changes over time in an accelerated manner (a rate of change of rate of change) it would have exactly the labels as does G. Thus, a close examination of what our math is really saying seems to indicate a new physics parameter: the acceleration of the specific volume of mass. Perhaps we should look for this characteristic of mass!

There is a confusion even here in the real (mostly) world of physics. Obvious to all is an area ten feet long by ten feet wide is 100 square feet in size. This seems to be 10 ft * 10 ft = 100 ft^2, or 100 square feet. But how, exactly, does one multiply by feet? A better description is that we have one row of 10 square feet times 10, or 10*10sq. ft. = 100 sq. ft., or perhaps, 10 sq.ft./row * 10 rows = 100 sq. ft.

A multiplier is an operation resulting in the *addition* of several multiplicands of equal size. An exponent is an operation resulting in the *multiplication* of several bases of equal size.

4 * 3 = 3 + 3 + 3 + 3 = 12

3^4 = 3 * 3 * 3 * 3 = (3*3)*(3*3) = (3+3+3)*(3+3+3) =

(3+3+3)+(3+3+3)+(3+3+3)+(3+3+3)+(3+3+3)+(3+3+3)+(3+3+3)+(3+3+3)+(3+3+3) = 81

Notice that the multiplier and the exponent, the 4 in both cases, does not appear in the calculations of the result! It is only a count. Also note that with exponents the roles of multiplier and multiplicand are no longer defined.

Rather disturbing is the use of multipliers and exponents which are not drawn from the set of positive integers; they are counters yet we allow counting to non-integer numbers such as 1/3 and *pi*. How does one count up to *pi*? Analytic algebra is an area of math that makes great use of techniques allowing this ‘counting’. Many functions don’t cover a complete domain and analytic algebra finesses this by creating (sometimes only with great difficulty) an analytic continuation of the function; an extension of the *function* that can be applied to the entire domain of whatever set of numbers is desired.

A simple example of this is the natural logarithm function, ln(x), x > 0. To cover the entire continuum of possible real number inputs (not just x > 0) Euler’s exalted equation can be used: $e^{\pi i} = -1$. Taking the log of both sides allows for x to go negative: ln(-1) = π*i*, which can be expanded to include all negative numbers. A much more complicated example is the famous Riemann zeta function which only covers real inputs greater then 1, but can be analytically continued to cover all the reals.

A quick look at the above equation shows e, an irrational number, being multiplied by itself an [irrational number multiplied by an imaginary number] number of times. This procedure is an extension of the *operations* of exponentiation and multiplication beyond counting to something else entirely.

Multiplication is a kind of iteration of addition. There is a single quantity to be ‘added to itself’ a certain number of times. This is a rather misleading way of stating what actually occurs. A single quantity is added to a running total starting at zero (starting with ‘no-thing’) a certain number of times. For those of us who don’t like zero as a number it can be expressed as a single quantity added to a running total starting with the single quantity for one less than a certain number of times.

Earlier in this section we asked how one counts up to a non-integer number. The analytic extensions that allow this for multiplication and exponentiation depend upon distributive laws. For example, the product of 2 ½ * 6 stones is really (2 + ½) * 6 stones which is 2 * 6 + ½ * 6 stones which is 12 + 6/2 or 15 stones. So a non-integer multiplier is another form of well hidden uncompleted operations, this time of addition and division. Note, ½ * 6 still does not make sense, as 6 added to itself ½ times, or any of the other ways of expressing the operation, is not a defined operation. Only when the symbol ½ is viewed not as a number but as an uncompleted operation of the two numbers 1 and 2 can the operation be attempted.

This example can be used to show the mind set of ‘continuous and infinite’. For example, there are two ways to visualize what happens when 5 stones are multiplied by 2 ½. The first is that we see stones moving from one big pile into a new pile one stone at a time. Every time five stones are moved we add one to our count. When 10 stones have been moved we have a count of 2 and we now need to move a total of 2 ½ (½ times 5) more stones. We move 2 more and then when a stone is still between the two piles we split it in half and put only half a stone in the new pile. This assumes we can not only split the piles of 5 stones but that we can split a single stone as well.

The second visualization is that we have a number of piles of stones, each pile having 5 stones. We attempt to draw a circle around just 2 ½ piles and immediately run into problems: it cannot be done.

Exponentiation by non-integers is also a form of distribution. For example, $5^{2.5}$ is really the product of $5^2 * 5^{.5}$ which is 25 * √5, hidden operations of multiplication and square roots.

In any event, the ‘certain number of times’ is not a quantity of some object like stones. As noted above, multiplication is a different operation for each multiplier, but not so exponentiation, and this fact is exemplified by the sigma and pi operators.

Sigma, Pi, and Integration Notations:

These three mathematical symbols (∑, ∏, ∫) demonstrate the iterative count functionality of multipliers, exponents, and integration limits.

Looking first at a typical sigma symbol we see:

$$\sum_{n=1}^{\infty} 1/n^2 \quad = \pi^2/6 \quad \text{(‘in the limit’)}$$

The ‘n’ is the counter, which in most applications of the sigma notation also appears somewhere in the expression to be evaluated. The count goes in integers from 1 to ‘infinity’. The resolution of this sigma notation can be given because the series ‘converges to a value at infinity’. But we can formulate a different, simpler, example of the sigma notation:

$$\sum_{n=1}^{4} 3 \quad = 12$$

Here we are adding 3 ‘to itself’ 4 times – in other words multiplying 3 times 4. Another example:

$$\sum_{n=1}^{10} 3 \quad = 30$$

Now we can see that the multiplier is actually part of the unary sigma operator, not part of a binary operation. These last two examples show exactly what multiplication is: repeated addition a certain number of times.

The first example above includes the counter (index) part of the operator in the expression to be operated on, which does complicate and confuse the concept. Taking a simpler example than the first but keeping the counter in the equation:

$$\sum_{n=1}^{4} n \quad = 10$$

we see a kind of ‘additive factorial’ operation, where the counter itself is the subject of the addition. This qualifies as one of Hofstadter’s ‘strange loops’, however it is so common as to be overlooked. When the indexing does not start at one, the nature of the counter is even more apparent:

$$\sum_{n=7}^{10} 3 \quad = 12$$

This form, and that of pi notation and integration notation, is very useful and offers great flexibility and compactness to expressions such as Riemann's zeta function:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{, s a complex number,}$$

but at the expense of confusion at a rather subtle level.

The integral symbol, like the sigma symbol, is meant to convey a sense of 'sum', and is an elongated letter 'S'. It, too, has indexing indicating the limits of integration, which are a kind of counter. Integration will be seen again in Section 7.

The pi notation has very similar features as the sigma notation:

$$\prod_{n=1}^{4} 3 \quad = 81$$

but now the counter acts in place of an exponent, in this case 3^4. However, whereas the sigma argument can be a labeled quantity, the sum of 10 stones, the pi notation can not be a labeled quantity. It makes no sense to raise 3 stones to an exponent.

Notice in both sigma and pi notations the counter's range is the upper limit minus *one less than* the lower limit. This 'one less than' is a symptom of the 'slabs and cracks' confusion addressed in Section 7 as well.

These types of operators demonstrate the essential difference between a quantitative number, 3 stones, and a counter or index number indicating the repetition, range or limit of an operation. They also indicate another area where an extension has blurred important distinctions of number theory.

Section 5: Place Value and Base Number Systems

Encoding Information within Number Symbols:

Most are familiar with our base ten Hindu-Arabic number system, but few know of its strengths, or its weaknesses. Its most worthy attribute is that it allows humans with finite and limited powers of memory to do unlimited calculations. Its most worrisome aspect is the inclusion of zero as a digit. (The next topic in this section describes number systems without zero.)

The most basic use of any number system is counting, which is the comparison (mapping) of a well established sequence of unique symbols with items in the real world. The use of fingers (digits) for this task is thought to be a world-wide first step in counting. The base ten place value system is defined and named for the powers of the multiples of each 'place' or column of simple symbols. A simple symbol (as defined in Section 1, Conventions) is a unique symbol made up of just one element and occupying just one place or column. The base ten place value system has, by common standards, ten simple symbols: 0 through 9.

At its inception the base ten system probably had 9 unique simple symbols and used a blank space indicating an empty column. Multiple empty columns were difficult to interpret (this is well before typesetting!) so a mark, a dot actually, was used to indicate nothing in that place. The dot became a small circle, the circle a full sized oval, the oval a number, and zero was born. (There are other explanations of the evolution of the symbology of zero.)

Any number larger than 9 cannot be expressed as a simple symbol, but the columns with their powers of ten multipliers allows for compound symbols of unlimited size. The difference, the really big difference, between simple number symbols and compound number symbols is that you must *memorize the order* (relative size) of *only* the simple symbols, the *sums* of all binary combinations of simple symbols, and the *products* of all binary combinations of simple symbols. Once that is accomplished the order, sums, and products of compound number symbols can be easily computed, by humans, without any additional memory effort.

We all know that 7 is larger than 4, and that it is larger by 3. The sum of 7 and 4 is 11, the product 28. These facts we've memorized. But if asked which is larger, 315 or 522, we know the answer without needing to have memorized the order of integers out past the hundreds. (We have been taught to look at the 'longer' number as the larger, unless they are the same length, then the number with the largest first from the left digit, etc. Thus, 134 is larger than 97, but smaller than 221 or 145.) We can calculate the sum and product without problem or memorization. This is the true power of a base ten (and of any positional) number system.

The size of the base is crucial to its usefulness to human calculation. Some cultures used other bases such as 20, or 60, and computer scientists use binary, octal, decimal, and hexadecimal interchangeably. The larger the base the larger the quantities that can be expressed in a given number of places, and an adroitly chosen base with lots of divisors (like 60) can greatly ease common division problems. However, the larger the base the greater the number of unique simple symbols must be memorized as well as their order, sums, and products. If a base one million system were in use and an example of two of its simple number symbols were '▲' and '■', the questions of which is larger, what is the sum, and what is the product, would require the memorization of over one trillion facts about the simple symbols. Not something humans could accomplish. Our base ten system requires school children to learn a little over 100 such facts.

As to that pesky ‘number’ zero, which slipped in the back door of our number system, it must be relegated to the class of objects that are not numbers, as it can not express a quantity that is the result of real world calculations. Zero may be, as negative numbers may be, a useful intermediate step in calculations, but is not valid as data input or as a result.

If you have exactly two goats and sell both to your neighbor you ‘have’ no quantity of goats, that is, you do not have any goats. This situation can *not* be expressed numerically. ‘Zero’ is precisely that *state* (not number) of having no quantity of some item, much like infinity is considered to be not a number but a state of ‘more than finite’, whatever that means.

Also, not having any goats is not a relative condition. The speed of a ball tossed inside a moving train is relative to the observer: it appears to move faster to someone watching the train go by then it does to the people inside the train. The same can not be said for goats – they are not relative. Having 2 goats, then selling 2 goats leaves you in a different state ‘of goats’ than having 3 goats and selling 2 goats!

Another example: zero as an exponent has been defined to always result in 1, so that expressions such as the following (representing an 8 digit binary place value number) yield useful results:

$$\sum_{n=0}^{7}(a_n)*2^n$$

We can certainly do without zero even as an index, as shown by the equivalent representations:

$$a_0+\sum_{n=1}^{7}(a_n)*2^n$$

or

$$a_1+\sum_{n=2}^{8}(a_n)*2^{n-1} \quad .$$

These representations are not as compact as the first, but do not require special definitions for dealing with zero. Also, the last one points to the fact that the first place digit (the rightmost) is not multiplied by the base to any power, while the first, compact form, seems to indicate the first place digit is multiplied by the base to some power. The last form also correctly labels the position of the digits from the right *starting with one.*

Our decimal system is a number system based on ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. These ten symbols are used in combinations (with a few supporting symbols such as: ‘.’, ‘…’, ‘-’, and others) to express all of our numbers, including integers, rationals, irrationals, algebraics, transcendentals, etc.

Counting of a collection of objects starts with ‘1’ and continues along the memorized ordered list of digits, both symbolically and verbally. It wasn’t always so. For perhaps two million years the human species made do with ideas of ‘more’ and ‘less’ and perhaps with comparisons to ‘number of fingers’ displayed. This was developed into reckoning by tally, scratched lines on a surface. There is evidence of counting at least as far back in time as 50,000 years ago. The accompanying verbalizations to these efforts are lost to us. (The very abbreviated history given here, with its rough dates, is meant to illuminate mathematics, not history.)

Written numbers didn’t wait on the invention of paper; carvings, impressions (e.g., in clay), and tokens were in use in the region of present day Iran by 4000 BCE. Greek number symbols were used by the Greeks, of course, and Roman numerals arrived on the scene with the spread of Latin across Europe with the Romans, naturally. These systems were still just symbols for specific values used in conjunction with each other to express quantities, not really much different from tally marks.

With the rise of civilizations came a need for calculations with quantities. Land use, taxes, banking, apportionment, and accurate time measurement all required not only statements of quantities but arithmetic manipulations of those quantities. The example of a difficult number system is often given by attempting sums with Roman numerals, and while it may not reflect historic reality it certainly drives home the need of something better than just names for all the possible quantities.

We need to follow another historical thread to understand what a place-value system entails. All number systems have a symbol for the quantity ‘1’, most of which look like a single raised finger (index, not middle). Many systems also have a symbol for one hand of fingers and for two hands of fingers. The Romans used I, V, and X for those quantities. Early systems did not care about the placement of the various symbols in an expressed quantity: using Roman symbols but *not* the Roman number system, IIVXXXI would have been the same as XXXVIII (38 in our system). With the Roman number system (and many other systems) placement did matter to a certain extent. The quantity 6 was expressed as VI, the quantity 4 was IV. Notice that the symbol ‘I’ stands for the quantity 1 no matter were it appears in the number, but if it precedes a larger digit symbol it is subtracted from that symbol and if it follows it is added.

The Babylonians, well before the Romans, took the role of the position of a digit within a number to an advanced, true place-value system. In the middle of the 18^{th} century BCE the Babylonians, who used a 60 based number system, started letting the value of a digit depend not only on its symbol but where it appeared in the number as well. This is how our decimal system works: the number 303, for example, has two digits with the symbol ‘3’, but the first digit of 3 (from the right) has a value of 3, while the third digit, also a 3, has a value of 300. Without a zero digit the number ‘303’ in our decimal system would appear as ‘3 3’, with a space indicating there is no quantity of tens, just 3 ones and 3 hundreds. Since all this development was taking place before typesetting, differentiating between no space, one space and three spaces was difficult so a symbol (*not a digit or number*) was inserted to indicate a space. In those early place-value systems a separate digit symbol was needed for each integer up to the one less than the base of the place-value system. Most systems at that time were base ten which explains why these place-value systems have only nine separate digit symbols: a symbol for 10 was not needed as that was just a 1 in the tens’ place and a hard to see space, or later, a special symbol, in the ones’ place.

So we’ve come to place-value systems with nine digit symbols and a symbol for a space to make reading the number easier. The really huge advantage of a place-value system is that the symbol for any number is much more than just a symbol, it encodes within it how to use it in calculations. *This means that a place value number is really an uncompleted operation.* To explain this consider only the first nine integers in our decimal system of number. What is 6 plus 3? Just looking at the symbols ‘6’ and ‘3’ there is no way to calculate the answer. We are forced to memorize, as children, the addition and multiplication tables for the first nine (or 10 or 12 depending on the whim of the teacher) integers. The question 6 + 3 = ? is solved from memory: the answer is 9. But examine another question: what is 14 + 374? Don’t have this memorized? That’s OK, this expression can be restated, just by looking at the two number symbols (equations or expressions) ‘14’ and ‘374’ as the sum of:

(10 + 4) + (300 + 70 + 4),

which stated using the position values become the sum of each column:

3, 1 + 7, and 4 + 4,

which you do have memorized. When this set of calculations is reassembled it becomes 388, all done from just the nine digits' addition table.

Encoding information about the integer within the integer's symbol is a massive advantage to calculations. Borrowing and carrying became well understood and a mainstay of place-value computations. This is all done with just 9 digit symbols plus one symbol indicating space, used just for convenience in reading. So why did we need to add a tenth digit, zero, and what does it really mean?

In India Brahmagupta started using zero as a number in 628 CE, giving rise to the Hindu-Arabic system of numbers. This was spread to Europe by Gerbert of Aurillac in the late 10^{th} century and by Leonardo Fibonacci in the start of the 13^{th} century.

This new system was revolutionary in two ways. Zero, which started out as just an extra space between digits became both a digit and an integer, and, extensive use of a positional notation of digits, i.e., a place-value system became the norm.

The very start of an immense genre of mathematics was born when zero became a number. That genre is today called analysis. Most mathematicians place the start of analysis with the invention of calculus in the 17^{th} century, but in its most basic form it arises with the need of what we now call inverse functions or pre-images. If I have 10 sheep and I sell you 3, you add 3 to the number of sheep you already had and find the total sheep you now have. I, on the other hand must subtract 3 from 10 to find I now have 7 sheep. Subtraction is the inverse function of addition. Addition is closed for the set of positive integers; any integer added to any other integer yields an integer result. This is not so for the inverse of addition. Suppose I had only 3 sheep to begin with, when I try to subtract 3 from 3 I'm in trouble because there is no positive integer answer. Worse, if I have only 2 sheep and sell you 3 we're both in trouble.

To solve the first problem we simply let the symbol that stands for a space which means 'nothing' be a number and include it with the positive integers. We now have ten number symbols (actually 11 as we still have the space), and zero can now stand alone indicating a quantity of none. (This makes no sense. If I ask 'do you have any money?' and you respond , 'yes', and I then ask, 'how much?' and you respond, 'I have zero money', you are using zero as a numerical quantity, but do you really have any money?)

Solving the second problem is a bit more convoluted: we add another whole bunch of integers to the positive integers plus zero: the negative integers. When I subtract 3 from 2 sheep I obtain -1 sheep and say 'I owe you 1 sheep'. An interesting question arises: if I have -1 sheep and you had 6 sheep originally, how many sheep do you now have? You have 8 sheep plus ~1 sheep, where ~ means imagined, or delusional, or future, or perhaps in this case inedible.

Zero and negative numbers are extensions to the positive integers to allow arithmetic calculations to 'always work', that is, to allow the operation of subtraction to extend across the entire domain of positive integers (and, of course, with analytic continuation, over the new non-positive integers as well). This is a great convenience to accountants, both current and ancient, as they compute profit and loss, income and expense, and inventory over time periods.

As subtraction is the inverse function of addition, division is the inverse function of multiplication, and it leads to even more extensions.

Multiplication is closed on the positive integers. Its inverse, division, is not. One divided by four can be expressed as ‘what times four equals one?’ for which there is no integer answer. As with the negative integers a new set of numbers must be invented to allow division to be closed for all integers (positive and negative) as well as the new numbers themselves. These new numbers are called the multiplicative inverses of integers and the inverse number of the integer 4 is ¼. There are several problems with this analytic extension to numbers, problems even more serious than those of extending into the negatives.

First, note that the inverse numbers are formed by dividing one by the number to invert. The symbol reflects this uncompleted operation. So what number answers the question ‘what times zero equals one’? The answer is the inverse of zero, of course, which is 1/0, but that symbol is deemed to be ‘undefined’ by the mathematical community.

Second, another extension, this time to the place-value system, allows for places to have multipliers of these inverse integers. In base ten they are: 1/10, 1/100, 1/1000, etc. generally expressed as negative exponents of 10: 10^{-1}, 10^{-2}, 10^{-3}, etc, as well as an exponent of zero ($10^{0} = 1$?!). This raises two problems. In any base there are some inverse integers whose expressions in the place-value system have ‘infinite’ numbers of digits: 1/3 in base ten (0.333…) for example. It also allows for any numerical value to have more than one representation, generally a bad thing in a number system. Examples include 3.0 = 2.999…, and 123.0015 = 123.0014999… .

Third, this destroys the concept of two numbers being adjacent, as now, ‘between any two numbers there is a third number.’ Thus, a la Cantor, we have an infinite number of an infinite number of numbers. That means there are more real numbers than rational numbers (yet – between any two real number there is a rational number).

Calculation with numbers that have an infinite number of digits is not well defined, and leads to numbers that are not only composed of an infinite number of digits but do not ever repeat a digit or string of digits endlessly as does 1/3 and 1/999 (0.333… and 0.001001001… , both of which are more correctly expressed as $0.\overline{3}$ and $0.\overline{001}$, but that form is harder to see using computer fonts). A non-repeating decimal fraction number is called irrational, as it cannot be expressed as a ratio of integers. A bit stranger, some irrational numbers are also transcendental (not roots of rational polynomials). Three of the most famous irrationals are also the most useful: √2, π, and *e* (1.41421… , 3.14159… , and 2.71828… , here expressed correctly with ellipses, with π and *e* examples of transcendental numbers).

Decimal fractional notation, that is, the part of a base ten number to the right of the decimal point often called the decimal part, presents additional problems. Examining numbers with decimal parts we see there are three distinct types: terminating decimals (or ending in an infinite string of zeros or spaces to the right), repeating infinite decimals, and non-repeating infinite decimals. The first are always expressed without the trailing zeros: 123.45, and the second, as noted above, should be expressed with overlining: $123.\overline{45}$. The third is often expressed using an ellipse: 123.45… meaning ‘we haven’t calculated what comes next’. The first two types can always be expressed as a rational number: 123 and 45/100 for the first, and 123 and 45/99 for the second. As such, they represent an uncompleted operation, several in fact. 123.45 means multiply 100 by 1, add it to 20 multiplied by 2, add that to 1 multiplied by 3, and that to the fraction 45/100 by using common denominators, giving the still uncompleted operations of 12345/100. The third example: 3.1415… is uncompleted in the sense that it cannot *be* completed, but is better classified as not a number!

None of these extensions to numbers or systems is incorrect; all are, given the restrictions on zero, consistent. And most telling, our numbers and number systems are exceedingly useful. But the extensions are misleading and obfuscating. There must be a better number system to apply to the discovered laws of physics, biology, chemistry, and the sciences in general.

So, let's start over. When place-value systems were first introduced the number of digits needed was always one less than the base. We had symbols for only 1 through 9 for base ten and 1 through 59 for base 60. For those places that weren't needed a space was left: one hundred and 3 was written as just '1 3'. Zero crept into the picture as a symbol representing the space for ease in comprehension of the written number: '1○3'. So zero happened by accident – what if it had been better thought out before adding another symbol that would become such a thorn in the side of mathematics? Suppose our base ten place-value number system had started out with ten digits: one through ten (1 - #, using # as a symbol for the value ten)?

Please note, mathematicians are so enamored of zero that they have created such a system as described in the last sentence of the preceding paragraph and labeled it a 'bijective numeration system of finite strings of digits' but then, for the love of zero, they defined it as bijective to the *non-negative integers*. This means that while there is no symbol for zero, zero is, somehow, included in the a listing of the bijective numerals. How, you might ask? By including the 'empty string' as an element of the bijective numeration! Of course, one could easily miss this string in a list of the first 4 such numbers:

1
2
3.

Oh, well, we could just create a symbol for the empty string, perhaps {} from set theory, or wait, maybe we could use the symbol '0'! Actually, the Wikipedia entry for bijective numeration, when wanting to list the first few numerals, states, "using λ to denote the empty string". Surprise! yet another symbol representing no symbol.

The system created in this paper is not the standard bijective base-10 numeration system. It does not have a symbol for zero, it does not include 'no symbol' as a symbol! *It is bijective to the <u>positive</u> integers*.

<u>Systems Without Zero:</u>

It is certainly possible to form a base ten system with powers of ten place multipliers and using ten unique simple symbols, *but without a zero or 'place holder' symbol*. Using the first nine familiar symbols of 1 through 9 and adding the symbol '#' to indicate ten, we have just such a system. We count from one: 1, 2, 3, 4, 5, 6, 7, 8, 9, #, 11, 12, … , 19, 1#, 21, 22, … , 29, 2#, 31, … , 99, 9#, #1, #2, … , ##, 111, etc. New counting words might include tenteen, twenty-ten, ninty-ten, tenny-one, and tenny-ten.

Addition, multiplication, and all other arithmetic operations are just as before, with no more memorization required than for our normal base ten operations. There are several differences, however. 'Zeros' cannot be added to the left of a number, nor to the right of a decimal number, or appear within a number. The number of columns (places) a number requires are exactly the number of columns that have a symbol. This precludes the use of decimal numbers, at least those smaller than '.1' as '.01' does not exist in this system. Another consideration is that while # can not be expressed as 10, the number .# is actually 1, leading to multiple representations of the same number, a condition to be avoided in a well formed number system. Hence, this system can have no decimal (fractional) number representations, that is, it needs no radix point.

It is the thrust of this paper that a number system using only the positive integers to describe discrete finite aspects of the universe would not have need of decimal fractions as the objects being manipulated (stones, gold atoms, etc.) have a finite smallest unit (one stone, one gold atom) and if that scale is used there could not be a decimal amount expressed that related to the real world. A decimal fraction is actually an uncompleted division (see Section 1 Real Numbers and the previous section*).*

Place Value Numbers are Uncompleted Operations:

A rather subtle difference between the standard base ten number system and a base ten system without zero stems from a number such as 123 being an expression, that is, an uncompleted operation. This is best shown using subtraction. In the standard system, a grade school student might solve 123 minus 12 by saying (and writing), '3 minus 2 is 1, 2 minus 1 is 1, and bring down the 1, so the answer is 111'. She might have said, for the last calculation, 'nothing from 1 is 1'. What either means mathematically is that the space to the left of the minuend 12 is really a zero: 123 minus 012.

In a PIM system the number 12 has exactly two digits, the spaces to the left and right of the number can not be considered part of the number and zero (or any symbol) cannot be placed to the left of the number. What, then, would the student do and say when she comes to the third position of the subtraction problem? 'Bring down the one' works but it should be interpreted as 'no subtraction from the one'. Mathematically we must realize that the number symbols 123 and 12 are both equations (expressions): 1 * 100 + 2 * 10 + 3, and 1 * 10 + 2 (uncompleted operations); meaning 100 + 20 + 3, and 10 + 2. The operation of subtraction is actually only defined on positions that have digits in both numbers, so we have: 100 + 20 – 10 + 3 – 2, or 111, which is itself an uncompleted operation: 1 * 100 + 1 * 10 + 1.

As an aside, a more complex subtraction example demonstrates the PIM subtraction operation, which is defined only for subtracting a smaller number from a larger number.

12#3
 123
117#

Verbally this is, '3 from 3 means borrow ten so three from 13 is #, 2 from ten but we borrowed one so 2 from 9 is 7, 1 from 2 is 1, and no subtraction in the fourth place is 1, so 117#.

Computer Design for Positive Integers Without Zero:

The design of a computer that uses a binary number system without zero is certainly possible, and building such a virtual system may be quite interesting. The binary numbers stored in the computer registers would still have only two states, on or off, but there are several ways to adapt this to PIM math.

One way would be to let the bits represent 2 or 1 instead of 1 or 0. In the normal binary system the zero does dual duty as both a simple symbol and a place holder of no value. In the non-zero binary system there is no place holder so each number would require two registers (of fixed length): one for the digits of the number and one for a mask indicating the number of digits in the number. This requirement points to the fact that written numbers in a system with zero as a digit always have effectively an infinite number of digits: zeros (actually the equivalent: spaces) go to the left ‘forever’, as well as to the right of a rational decimal number following the decimal part.

While needing two registers for each positive integer number doubles the number of required fixed length working registers it may speed up calculations. For example if a 64 bit register contained a number that used only the last 5 bits only those bits would need to be considered by the logic of the operation.

But by far the easiest technique to build a PIM computer would be to make no changes except to make a register of all zeros an error condition indicating, perhaps, an underflow. Another definition of an all zero register could be NaN, which in many computer languages means ‘not a number’. It would indicate a state of ‘not having any’ as opposed to ‘having none’.

In any event, a positive integer without zero computer is certainly an attainable goal, should it become of value. For now, current computer technologies are very able to handle positive integer only computations while still using zero at the bit level base number system.

Section 6: Infinite Expressions

Infinity and Zero:

Mathematicians generally agree that infinity is not a number, yet it is used as a counting limit as if it were. It is also generally acknowledged that zero is a number, yet one must constantly guard against using it in the 'wrong' situations. A logical position would be that both either are or are not numbers, and as the thesis of this paper is that a finite mathematics may be useful, neither are regarded as numbers herein.

While infinity is not considered a number, expressions that involve an infinite number of operations are quite in vogue. Calculus, of course, is a prime example and caused an uproar of dissension when introduced for its use of infinitely large and small quantities. Seemingly resolved by limit theory, the problems are still there, just better hidden.

The derivative function is taught by using an increment that is eventually set to zero. For example, the derivative of $f(x) = x^2$ is computed by calculating $f(x+h)-f(x)$ divided by $x+h – x$, which for this example is $(x^2 + 2xh + h^2 – x^2)/(x+h – x)$ or $(2xh + h^2)/h$. At this point we must *not divide* by h, or worse set h = 0. Instead we *cancel* the h obtaining 2x + h, and only now, at the end, do we set h=0.

Do we consider the expressions (x-1)(3x+2)/(x-1) and (3x+2) to be equivalent? Well, yes and no. The first must not let x=1 be in the domain while the second is fine with x=1. Yet if we cancel the x-1 terms and then set x=1 we have replicated the process of finding the derivative detailed above and arrive at a possibly incorrect solution.

Limit theory replaces the above process with the idea that we let h be a positive number and calculate the sequence of results as we reduce the value of h towards zero. We then examine the sequence of results and determine if it is convergent, that is, if it seems to be 'going somewhere' both finite and fixed. Unfortunately, we would have to examine an infinite number of terms in the sequence to be certain it converged. Delta-epsilon methods can demonstrate closeness to a limit but it is not an exact solution, only an engineering 'as close as you need precision' solution, and even at that, it can not handle many types of sequences and functions.

Convergent infinite sequences, separate from calculus, are another very common use of an infinite number of operations. Again, limit theory comes to the pseudo rescue. Any reiterative operation, such as the sigma, pi, and integral expressions examined in Section 4, that have infinity as an upper limit are at best an uncompleted operation, at worst a form of best guess in an extended number system. Those expressions are now taken not to mean an infinite sum or product, but rather, the limit of partial sums or products.

All of this confusion and complication because we allow zero (and infinity in practice) to be a number. Zero exists on the continuous and infinite real number line but not in the real world. *Real world discrete finite quantities have as their least amount the value 1 at their inherent smallest scale.*

The rational numbers fare no better then the irrationals. The number defined by

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

is accepted as $\pi^2/6$. But it is not - it is, perhaps, the largest rational number less than $\pi^2/6$ (an impossible object by definition). Summing to infinity, or 'going to the limit' is obviously not possible, and equations representing this operation can not be completed. Looking at the sequence of partial results for n < 10, n < 100, n < 100,000,000, etc. we see that they are all rational numbers that get closer and closer to $\pi^2/6$. 'Infinitely close', when we sum to infinity, would seem to mean that there are no rational numbers greater than the infinite sum and less than $\pi^2/6$. But we know by conventional mathematics that between any two numbers there exists a rational number – another conundrum.

Worse, while the sum of any two rational numbers is a rational number, the sum of an infinite number of rationals may not be. A telling example is the decimal expansion of *pi*. Any decimal number may be seen as a sum of rational numbers: 2.21 is just 2/1 + 2/10 + 1/100. *Pi* is 3.1415… , an infinite sum of rational numbers, yet we 'know' that *pi* is irrational. Apparently every irrational number is the infinite sum of rational numbers. In contrast, the infinite sum of 0.333… is rational: 1/3.

<u>Taken to the Limit:</u>

An interesting way to determine (approximate) the ratio of a circle's circumference to its diameter is to draw regular polygons with progressively more sides (always an even number) and calculate the perimeter divided by the diameter measured from one vertex to the opposite vertex. The more sides the more accurate the approximation to π. Taken to the limit of an infinite 'number' of sides the exact value of π emerges. However, an interesting thing happens to the interior angles between the ever more numerous sides: each goes to 180 degrees in the limit! Hence, π is the ratio of circumference to diameter of a 'circle' with infinite sides but finite circumference and diameter. And much worse, the sides are all in a straight line of finite length but no end points!

<u>The Ellipsis and Overlining:</u>

One of the most common (seen in most repeating decimal and all irrational numbers) yet least rigorously defined symbols in mathematics is the ellipsis …

The best definition might be 'and so on': 3.333… and 3.1415… are typical examples. The first is actually better expressed as $3.\overline{3}$ and the second must be left as is. But due to the lack of definition many questions arise concerning the use of either form of 'and so on'.

One of the flaws of our real number system is that not all (in fact none) of the real numbers have a unique representation. The well known and confusing statement that $0.\overline{9} = 1$ is the iconic example. Of course 1 is the multiplicative identity so every number can be multiplied by $0.\overline{9}$ without changing its value. Multiplying by a number that has infinitely many digits is a bit of a problem and certainly represents an uncompleted operation, but nevertheless it demonstrates that all real numbers have multiple representations. Not so, by the way, for operations within the set of natural numbers.

The ellipsis should not be used to represent repeating decimal digits, that is the task of overlining. Both symbols, overline and ellipse, are very poorly defined. The consensus seems to be:

Ellipsis: used after a few decimal digits (right of the decimal point) to indicate 'continues indefinitely and without pattern'.

Overline: used over one or more decimal digits and after, perhaps, one or more decimal digits (right of the decimal point) to indicate the overlined group of digits is to be repeated indefinitely in the same pattern.

But the question is this, if these symbols can be used at all why are they restricted to just these circumstances and definitions? It is both entertaining and informative to have a bit of fun with these two symbols but before we do we must examine a symbol never mentioned in mathematics but in use constantly, consistently, and pervasively: the space, or ‘no symbol’.

The Space and Zero:

When adding the two numbers 12 and 2, the twos are added but what is added to the 1? The answer is simple: verbally we say, ‘2 plus 2 is 4 and bring down the 1’, or we say, ‘2 plus 2 is 4 and 1 plus nothing is 1’. What we are doing is converting the space under the 1 into a zero. *Spaces are zeros.*

There are ‘unlimited’ spaces to the left and right of every written decimal number. The number 1 is ‘really’ the number … 01.0 … after converting all the spaces to zeros. (We really should indicate the number as $\overline{0}1.\overline{0}$ with an ‘and so on’ going to the left of a number as well as to the right. However the overline does not show up well in computer text, so often, incorrectly but without confusion, we use the ellipsis.)

It is important to recognize that this unlimited string of zeros to the left of a number is not a new idea or invention – it has always been such, just expressed as an unlimited availability of spaces.

Fun with the Ellipsis and Overline:

We have discovered in the preceding topic that the overline symbol can be applied to indicate the continuance of a pattern of digits ‘forever’. An expanded definition could be that an overline appearing to the right of a decimal point continues that pattern to the right indefinitely and an overline appearing to the left of a decimal point continues that pattern to the left indefinitely. Thus $\overline{3}.\overline{3}$ would indicate (stated improperly using ellipsis) … 333.333 … . But what, exactly, could that mean?

Taking a simple and familiar example can guide us to the meaning of these symbols, however, it is my opinion that any use of the ellipsis or overline to represent a number is at best misleading and likely incorrect. Nevertheless, we ‘know’ that $0.\overline{9} = 1$. This can be ‘proven’ by several demonstrations:

$1/9 = 0.\overline{1}$
$9 * 1/9 = 1 = 9 * 0.\overline{1} = 0.\overline{9}$
so $0.\overline{9} = 1$,
or by:
$x = 0.\overline{9}$
$10 * x = 9.\overline{9}$
$10 * x – x = 9 * x = 9.\overline{9} – 0.\overline{9} = 9$
so $9 * x = 9$
$x = 1 = 0.\overline{9}$
Using similar arguments we can demonstrate some strange ‘facts’:
$x = \overline{9}.0$ (note the bar over 9)
$10 * x = \overline{9}0.0$
$x – 10 * x = \overline{9} – \overline{9}0 = 9$
$x – 10 * x = 9$
$x – 10 * x = -9 * x = 9$
$x = -1$
so $x = \overline{9}.0 = -1$

Now we have $0.\overline{9} = 1$ and $\overline{9}.0 = -1$ therefore $\overline{9}.\overline{9}$ must be zero. And, since $10 * \overline{9}.\overline{9} = \overline{9}.\overline{9}$ then $\overline{9}.\overline{9}$ ‘has’ to be equal to zero. Interesting … but can we really define an infinite operation such as $9 * 0.\overline{1} = 0.\overline{9}$?

Turning now to the ellipsis we can ask (but not really answer) is there any use of the ellipsis to the left of a number? Pi expressed as 3.1415 … is universally understood in meaning. What would a number like this, … 723.8, possibly mean?

An improper use of the ellipsis in place of the overline is easier to understand in some instances, and using it we unearth many problems. In common usage 0.999 … is understood to mean $0.\overline{9}$. We can, however, expand the definition of the ellipsis (actually the overline) to allow it to appear anywhere within a number. This leads to some interesting questions.

Are the two numbers 0.999… and 0.999…999 the same numbers? We can map the first to the second bijectively so they have the same size (a la Cantor). We can also map them in a manner that leaves the trailing digits out of the mapping, which is why some might say the trailing digits are hyperreals. But the definition of same size does not limit the comparison to only one possible mapping; any bijective mapping takes precedence over alternative mappings that may not be bijective. As proof consider comparing the even positive integers to the positive integers. If both sets are compared in their natural order there is a bijective map. However, order the positive integers in this manner: even numbers first followed by the odd numbers. One way to map would be integers to evens but that would leave all the odds out of the mapping. But the integers are defined to be not more numerous than the evens.

Another interesting point concerning 0.999… 999: if we add to it the number 0.000… 001 we obtain 1 (using completely undefined addition rules for infinite digit numbers of improper type!). This would seem to argue that either 0.000… 001 *is* hyperreal (whatever that means) or that 0.999… is not equal to 0.999… 999. However, 0.999… 999 * 10 = 9.999… 999 and the above demonstrations apply as well: 9.999… 999 = 10 * x; 9 = 9 * x; x = 1. Which seems to argue the two representations are equal.

Given that spaces go on to the left of an integer, we can represent 1 as … 001. If instead we choose to represent it as 000… 001 we can use Cantor’s proof to demonstrate the integers are uncountably infinite. Given a listing of ‘all the integers’ in this new format, we can form an integer not on the list by making the first digit from the left of the new number different from the first number on the list, the second digit of the new number different from the second digit of the second number on the list, etc., thus forming a number different from any on the list. Thus, the integers, at least in this format, are uncountable.

The premise in all of this is that if you can use a symbol such as the ellipsis or the overline somewhere within a number then it can be used anywhere. If this use gives rise to many questions, even contradictions, it leads to the conclusion that the contradictions are also present in the standard usage of these symbols, just better hidden and commonly ignored.

Derivation of Real Numbers – Dedekind Cuts:

In the last part of Section 1 we examined how the negative numbers are incorporated into the integers with mathematical rigor, and determined that negative numbers are just uncompleted operations. Here we find the same situation with the real numbers.

Real numbers, at least the irrationals, can only be defined by what they are not, or by constructions composed of undefined quantities. A definition that uses an argument of ‘in the limit’ is at best an uncompleted operation, at worst a negation – a definition of what the object is not. All definitions of irrational numbers start with the rational numbers, which we have seen have problems of their own. Further, real numbers are defined using limits, but limits are defined using real numbers.

We assign a length to the hypotenuse joining two unit lines forming a right triangle, blithely assuming such a line exists with length the square root of two, an uncompleted operation. Pi is supposedly the ratio of a circle’s circumference to its diameter, again assuming that the two lines can co-exist and could be measured at all. (“Imagine a person with a gift of ridicule. First that a negative quantity has no logarithm; secondly that a negative quantity has no square root; thirdly that the first nonexistent is to the second as the circumference of a circle to the diameter.” Augustus De Morgan. This references Euler’s famous equation $e^{\pi i} = -1$, therefore $\pi = \ln(-1)/\sqrt{(-1)}$.)

Many definitions of irrational numbers have been advanced and accepted by current mathematicians. One such definition is the Dedekind cut. A Dedekind cut defines the irrational numbers. The cut partitions the *rational* numbers into two non-empty sets, A and B, such that every element in A is less than every element in B, and stipulates that A has no largest element while B may or may not have a least element. If B has a least element then *the cut* defines that rational number. If B does not have a least element then *the cut* defines an irrational number, since the cut cannot be in A or B as it would then be the greatest rational number in A or the least rational number in B, both of which have been stipulated to not exist. If it is not in either partition of rational numbers the union of which includes all rational numbers the cut can not be rational. Assuming that it *is something*, it must be irrational (meaning non-rational).

This defines the irrationals by stating they are *not rationals*. Actually the definition uses three negations: the set A does not have a greatest rational, the set B does not have a least element, and the cut is not in A or B; along with an assumption of existence – that some *number* exists at the cut.

Let’s examine the several problems with this definition of irrationals, as well as logical problems with the generalized theory of Dedekind cuts. Assume we have a number line of points and we pick one point on the line. Next we divide the rational points on the line into two sets: those strictly less than the selected point, set A, and those greater than or equal to the selected point, set B. Right away we notice that we have assumed that such a division can exist. We assume the existence of non-rational numbers to prove their existence. But moving on, set B must either have or not have a least rational number. If it does it must be the selected point, if not it must be an irrational number. But how do we know if set B has a least element? What is being stated is that if set B does not have a least element then irrational numbers exist. That does not prove their existence. ‘If elephants can fly we need strong umbrellas’ does not prove the need of cast iron umbrellas.

Applying the generalized Dedekind cut concept to the integers demonstrates both the problems with the cut and how a discrete mathematics eliminates the entire problem. Suppose we have a number line of the integers and we apply a cut to it. The selected point is greater than any point in set A and less than or equal to any point in set B. All of the above arguments apply to this construct as well, but we can do better than ‘a number line with points’. Picture a number line of integers, but instead of delineating points on the line, each integer defines a unit length on the line (see Slabs and Cracks, Section 7). The unit length is indivisible and finite, and the units are in the normal order with absolute abutment to their neighbors. That is, there is no space between the units; there are no points between the units or indeed, no points on the line, only discrete, indivisible, unit lengths. The lengths are what constitute the line, which makes more sense than points constituting a line. A Dedekind cut cannot fall on a point, let alone a point between units, as there are no points. A selection can only fall on a unit length. Note, it can fall *on* a unit length, not *in* a unit length. The unit lengths are indivisible, they have no beginning point, end point, middle point, etc. A unit length is selected or not in its entirety. To partition the discrete number line into two sets one must pick a distinct unit line (there is no other choice – there are no points between the lines). Then, if it is desired that the union of the sets constitutes the entire line, and that all lines in set A are less than all lines in set B, either A or B must include the selected line. No more problems, and no ‘numbers’ other than the integers represented on the line by discrete, indivisible, finite, unit lengths. This is the basis for Positive Integer Math and Quantum Calculus, described in other papers by this author.

However, while the unit length works well for discrete objects like gold atoms it cannot be taken to the extreme, as the next section explains.

Lengths, Physical vs. Mathematical:

In assuming that most, if not all, things in the universe exist in discrete, finite sized objects whose number must be finite we posit that these objects have an innate size (length, volume, charge, etc.) of one. Any measurement of a conglomerate of these objects, such as a bar of gold, would therefore be an integer number of innate units. A relevant question arises: is there a smallest unit of measurement? This might be seen as a smallest unit of space. This question brings into focus assumptions of Euclid’s geometry and just what it means to ‘measure’ something.

Euclid, before tape measures, needed to compare lengths of lines in similar triangles. Without expounding on his techniques we note that he was comparing one line to another line. All measurement is comparison. Today we compare an object to a ruler, assuming everyone has a similar ruler, which is just a standardized measuring stick. Without comparison a boy might say to another:

boy 1: “I have a stick!”.
boy 2: “How long is it?”
boy 1: “It’s as long as my stick.”

which tells the second boy nothing about the length of the stick. However, if both boys have sticks the conversation might go something like this:

boy 1: “I have a stick!”
boy 2: “So do I, how long is yours?”
boy 1: “Mines twice as long as yours!”

which tells the second boy quite a bit about the first boy’s stick. (Perhaps I should have not used boys, or sticks…)

Using innate units of measure, one gold atom or one stick, works fine: the ruler could be marked off in units of gold atoms or sticks. Of course, at the macro scale such measurements could be extremely cumbersome. Asking for a smallest unit of measure for measuring any and everything, besides raising questions of incommensurate sizes, also forces some logical inconsistencies in the fields of geometry, analysis, and number theory.

One must be very careful in ascribing characteristics to mathematical constructs. If an innate unit of measure of some real world object is a count, e.g. number of gold atoms, there can be no ‘half atoms’. If an innate unit of measure is a length, there can be no ‘points’ within the length, as demonstrated by the following description of inconsistencies that arise if this restriction is ignored.

Suppose we assume there is a smallest unit of length, and construct a number line made up of those lengths and numbered from the origin starting with 1. In lining up the units on the line we place them end to end – which implies each unit has two ends. To be specific let’s put a point at each end of each unit, and include the two points as part of each unit, assuming we were to allow *points* on a smallest *length*. Now we can ask if there are any points between the end points of a single unit. If there are then the unit would not be the shortest unit as it could be divided at the interior point. If there are no points within the unit then the two end points must be adjacent to each other. They can not be the same point as then the unit would not be a length, and they cannot have any points between them as then the unit would not be the smallest. So we have adjacent points – very non-Euclid!

If we allow adjacent end points for our smallest unit just what is its length? It must be greater than zero as they are distinct points, but as there is nothing between them and the points themselves have no length it seems to not have length. There is also the question of whether or not the end points of adjacent lengths are adjacent: they must be – but that means the length (distance between points) of each unit is the same as the length between units!

It might be fun to create a geometry with adjacent points. Several questions present themselves: Are there two types of relations between points: adjacent and not adjacent? How many points can be adjacent to a given point? Can a line be made up entirely of adjacent points? What happens to Dedekind cuts?

Suppose we construct a line of adjacent points and label each point with an integer starting with one on the left end of the line. We have a line with only the integer points on it. There are no other possible points: no Dedekind cuts are possible. Any point selected for the cut would always be the smallest integer in set B. Point 2 is adjacent to point 1, point 3 is adjacent to point 2, but point 3 is not adjacent to point 1 as point 2 is between point 1 and point 3. So we do have two varieties of relation between two points: adjacent and not adjacent. A subset of the line consisting of just points 1, 2, and 3, is 3 points in length, even though we can’t visualize the length of such a construction physically.

Measures of different dimension can not be mixed: no points (0 dimension) on lines (1 dimension).

<u>Fractals, Physical vs. Mathematical:</u>

Koch, Julia, Cantor, Menger, Sierpinski, and of course, Mandelbrot all worked with mathematical formulas that when executed ‘to infinity’ produced very strange results. Some of these results can easily be found on the Web, such as the Koch snowflake, Julia sets, Cantor dust, Menger universal curve, Sierpinski sponge, and the Mandelbrot set. These are all mathematical constructs but it was noticed, particularly by Mandelbrot, that some of these constructs, which he called fractals, took on the appearance of objects in the real world. Computer graphics designers were soon creating fractal mountains, fractal rivers, fractal clouds, and fractal coastlines. These creations were much less demanding of computer processing power and memory than using pictures of the real objects, particularly if three dimensions were desired.

Then mathematicians like Barnsley discovered how to mathematically describe, in a very few equations, complex objects like trees and ferns with amazing realism. It was hailed as the key to how nature created varied but similar complicated real objects, like trees and ferns, with only a few DNA blueprint like instructions. The Barnsley fern was (and is) a fascinating object to both watch being created and see the final result.

But there are huge differences between the Barnsley fern and a real black spleenwort fern, the variety most commonly replicated. Since all the above mentioned gentlemen were educated mathematicians, once they wrote out the formulas they did not hesitate to iterate them ‘an infinite number of times’. Of course, they really only iterated them sufficient to demonstrate what the end result would be if they did ‘go to infinity’, but the math paradigm of a real infinite allowed them to visualize nature iterating some simple set of instructions ‘forever’ to create real objects. This is very misleading in at least two ways.

The mathematical creations are of chaotic form; they create the picture in a point by point disjointed manner. Nature creates ferns in a growth sequence of connected processes. Simcha Lev-Yadun, in Plant Signaling & Behavior (2012 May 1) states, “The popular demonstration of drawing a mature fern leaf as expressed by Barnsley’s fractal method is mathematically and visually very attractive but anatomically and developmentally misleading, and thus has limited, if any, biological significance.” And, “A realistic set of mathematical equations to describe fern leaf or cauliflower curd development is needed.”

Beyond the botanical differences between the mathematical ‘theory of ferns’ and the physical ferns there is the falseness of the assumption that physical processes can be iterated like mathematical formulas. Any physical process must end with finite results in finite time. While a fractal coastline can be measured with ever decreasing units of measurement yielding different lengths and resulting in a non-integer dimension, a real coastline, if it could be frozen with, say, a very high resolution photograph, would have as its smallest unit a single grain of sand. A coastline would be a long string of single grains of sand, finite in both number and size. Reality does not allow infinite processes or numbers.

Section 7: Miscellany

<u>Slabs and Cracks:</u>

When counting, no one, except a computer programmer, starts with zero as the first counting number. Whether it's concrete blocks or fence posts the count starts with one. On the other hand, when measuring something everyone, except the carpenter who doesn't trust the end of his beat up tape measure, always starts with zero. Whether it's the length of a sidewalk or the length of a fence, the end of the tape measure is labeled zero (or should be, but most leave the label off). The two types of activities, counting and measuring, are very different in concept but actually become identical in a discrete, non-continuous universe: they are both a form of counting.

A typical suburban sidewalk is composed of slabs of concrete of the same length, separated by the surface grooves placed in the wet concrete to ensure that if the ground heaves under the sidewalk the concrete will fracture at the grooves, posing the least problems for pedestrians. If the sidewalk in front of your house were composed of 4 foot long slabs of concrete and you wanted to know how much (much → measure) sidewalk you had to clear of snow, would you go out and count how many (many → count) cracks or would you count how many slabs?

Counting the cracks you would start with one at the first crack; counting the slabs you would start with one on the first slab. If instead of counting cracks or slabs you used a tape measure with units of 4 feet (the '1' would appear 4 ft. from the end, the 2 at 8 ft. etc.) you would put the hooked end (called 'the dumb end' in surveying parlance) in the first crack and read the tape at the last crack. Since the tape starts with zero the conversions between the cracks measured by the tape and slabs is simply slabs = cracks. What actually happens here is the tape measure is a count of the tape's smallest increment, starting with zero. Survey tapes generally measure to hundredths of a foot, so your sidewalk might be 12,000 units (hundredths of a foot) long, as a lot size of 120 feet is common.

Counting slabs would yield 30 slabs, or 120 feet (30 * 4 ft). But notice that counting slabs you count real physical things, while counting (starting with 1) cracks or hundredths of a foot you count the divisions between the real things, and the divisions are not in themselves real. The conversion between the two is always slabs = cracks minus one. (Asserting that you only count the first and last crack as ½ crack each assumes you can divide an entity of zero size into two parts!) But while you can convert between them the two operations are different.

The surveyor's tape implies that the slabs can be divided, perhaps endlessly but at least from four feet to 1/100 foot. Counting slabs implies no such thing.

If measuring used the smallest discrete unit that the object being measured can be divided into, then measuring becomes counting, using the slabs = cracks formula.

This difference of one when counting plagues mathematical expressions. The sigma, pi, and integral notations must sometimes start with zero, sometimes with one. The same notations, along with the combinatoric equations must sometimes go to 'n', sometimes to 'n+1', and sometimes to 'n-1'. Programmers dealing with different languages must constantly check to see if a logic loop is executed the correct number of times, with concerns about whether it starts at zero or one, and goes to n, n-1, or n+1. (In programming talk these are OBOEs: 'off by one errors'.)

While subtle these irritations indicate an underlying problem with how we count and measure.

The Magic Number '*m*':

It is hard to determine whether the first misunderstanding of what should constitute a number occurred with the inclusion into the realm of legitimate numbers of zero or of the negative integers. But the foundation of almost all uncompleted operations called numbers (excepting only the rational numbers) can be traced to one magic 'number': '*m*', for magic.

Given just the positive integers and *m* we can re-create all the rest of the real and imaginary numbers. Here are some examples of the power of *m*:

for any positive integer 'n' and the magic number '*m*':

- $\pi = \ln(m)/\sqrt{m}$
- $(1 + m)^m \to \infty$
- $m*n = -n$
- $n^m = 1/n$
- $m^2 = 1$
- $m = e^{\pi i}$
- $\sqrt{m} = i$
- $1 + m = 0$

The last two bullets have certainly given it away; *m* stands not for magic but for minus one. With just this one 'number', which has no basis in the real universe, all the rest of the numerals can be constructed. The true culprit of confusion is not the irrationals, or the imaginaries, or the rationals, or even zero or infinity, it is the negatives. Specifically negative one from which all the rest are formed.

Cancellation vs. Division:

We all learn to 'cancel the top and bottom of like numbers' in grade school. We learn cancellation using the field of real numbers. For example, often when calculating fractions we have as intermediate results something like:

$\frac{2*x*17*6*y}{3*x*17}$ we then 'cancel' the x (hope it's not zero!), the 17, and the 3 into the 6 (=2*3)

giving $\frac{4*y}{1}$ or 4y. None of this is actually the operation of canceling – it is all division.

Fields have multiplicative inverses (division) so cancellation as an operation is not required. In these cases cancellation is division. Cancellation is defined in group theory and useful for algebras which have no defined division (such as the complex numbers where one multiplies by the fraction conjugate/conjugate).

As pointed out in Section 6: Infinity and Zero, canceling is used in place of division when division may be 'not allowed', that is, when the divisor may be zero. But this is not a correct use of the operation of cancellation.

In the real world quantities have labels (dimensions) and division or multiplication by a quantity with a label is not physically possible and should not be mathematically possible either. Cancellation is the only operation applicable. This is particularly obvious when dealing with ratios.

Ratios and Cancellation – the Arithmetic of Dimensions:

Quantities with like dimension (labels) can be added and subtracted: 3 apples plus 2 apples is 5 apples. Quantities with different dimensions cannot be added or subtracted: 3 apples plus 2 oranges is just 3 apples plus (or 'with', 'and', or ',') 2 oranges.

Quantities with *any* dimensions (or none) can not be multiplied, divided, or taken to a power by quantities with *any* dimensions. A multiplier, divisor, or exponent cannot have a dimension or label. How, then, to explain the success of the simple physics formula distance (feet) equals rate (feet/sec) times time (sec)?

More concretely, 10 feet = 2 feet/sec * 5 sec. Suppose we found the rate experimentally by measuring the travel of an object over 16 feet in 8 seconds. We 'divide' 16 by 8 and obtain the speed of 2 ft/sec. Next we 'multiply' 2 ft/sec by 5 sec to arrive at a distance of 10 ft. traveled in 5 seconds. This calculation is an example of the confusion among divisions, fractions, and ratios.

The rate as expressed above is not a division, it is a ratio. It is a ratio of two dimensions and as such any similar ratio could be substituted and give the same correct numerical value. The rate could be 2 ft/sec, or 4 ft/2 sec, or 16 ft/8 sec (although, there is different information in, for example, the two ratios 4/2 and 16/8, see below). When we 'multiply' the rate times the time we are really canceling a dimension of the ratio, while multiplying and dividing the pure numbers attached to the ratio. The expression of the ratio in the form 4 ft/2 sec can be written 4/2 ft/sec. When 4/2 ft/sec is multiplied by 5 sec we multiply the 4/2 * 5 to obtain 10 but we *cancel* the sec of the rate ratio with the sec of the time quantity to obtain just feet: 10 ft.

*A much better expression of the formula might be distance = distance:time * time, or for the example above: 10 ft = 4:2 :: ft:sec * 5 sec. More generally, x distance = y:z::distance:time * w time.* Notice that if we divide both sides of the equation by *5 sec* we obtain the proportion 10 ft / 5 sec = 4 ft / 2 sec, a valid expression, from an invalid operation.

The algebraic rules must be defined and fully stated for the manipulation of ratios and proportions, and of course, such an algebra must give the same results we obtain with the current concepts of fractions and labels, which have, obviously, proven to be quite useful.

The most common definition of *ratio* is an ordered pair of numbers which when viewed as a fraction indicate how many times the first number contains (can be divided by) the second number. Explicitly it's defined as either an ordered pair of numbers, or the fraction of the first over the second, or the value of that fraction expressed as single number. Thus, while a ratio may start out as the comparison of two integer values it can be expressed as a real number equivalent quotient. Euclid's definition 5 (of Book V) addresses this fact in dealing with the issue of equality of ratios, what we call proportions. Today we determine that 2/3 and 4/6 are equal ratios by finding that the quotients (viewed as fractions) are equal: 0.6666… . Euclid saw 2/3 as an incommensurate quotient (he did not use 'real numbers') so this was not a procedure he could apply. Instead, he created a definition that is the forerunner of the Dedekind cut argument, with all its attendant problems (see Derivation of Real Numbers – Dedekind Cuts, this paper).

A further part of the common (but not stated) definition of ratio is that the two numbers may have the same units, different units, or one or both be without units – 'pure numbers'. Additionally, ratios may be expressed with more than two numbers, such as 2 : 8 : 10 describing, for example, a box of those dimensions.

This definition of ratio is so broad that it completely obscures the nature of a ratio:
A *ratio* is an *ordered comparison* of two numbers, or units, or numbers with units, written as two entities separated by a colon. A *proportion* is a *comparison denoting equality of two ratios*, written as two ratios separated by a double colon.

Thus we have: 3:4, x:5, a:b, and 4:1 as numerical ratios and feet:seconds, inches:feet, velocity:time, and area:gallons as label ratios. Note that velocity is also a ratio, and that area of coverage by gallons of paint depends, as do many such ratios, on other considerations (thickness of the coat of paint).

Great care must be taken to define the two quantities of a ratio. For example, if a gambler wins 6 games and loses 2 his win to loss ratio is 6:2, but his wins to games ratio is 6:8. In the first ratio, 6:2, the total number of games is just the sum of the wins and losses. In the second ratio the sum of the two quantities has no particular meaning, rather, the difference reflects the number of losses.

Conventionally, ratios with the same dimension (label) become ‘dimensionless’ pure numbers. Also, generally accepted convention allows that ratios with different dimensions are ‘rates’.

The most familiar pure ratio is *pi*, the ratio of a circle’s circumference to its diameter. If we were to measure an actual circle with a diameter of 1 inch we would find a circumference of about 3.14 inches, thus making the measured ratio approximately 3.14 inches / 1 inches. Given 3.14 as a circle’s circumference to diameter ratio then a circle with diameter of 4 inches would have a circumference of 4 inches times the ratio 3.14 inches/1 inch giving 12.56 inches^2 / 1 inch, or, after the common practice of canceling labels, 12.56 inches. As per previous arguments about multiplying by labeled quantities, the ‘inches’ in 4 inches must *first* be canceled with the ‘inches’ in the denominator of 3.14 inches / 1 inches, *then* the multiplication by the pure number 4 can be completed. Using *pi* as a pure number constant by canceling its labels is a misleading shortcut to the correct answer. A circle with a 4 unit diameter has a circumference 4 times the circumference of a ‘unit circle’ whose circumference is 3.14 units, hence 4 times *pi* units.

Pi should be expressed not as a pure number but as a ratio of the same units. Using the term ‘unit’ to indicate any label, *pi* would be about 3.14 unit / 1 unit. *There should be no ‘pure number’ constants in physics.* (As explored later, *pi* may not actually be a ratio of the same units, as the two are incommensurate.)

The most familiar rate in physics is that of velocity, which is length per time, e. g. feet / second. In fact, every ratio that is not a same unit ratio (a ‘pure number’ ratio) should be reducible to a measurable rate of some kind. As an example, ‘big G’, the gravitational constant when viewed as a rate becomes much more understandable as perhaps not being a constant, but rather varying over time. G viewed as a rate is the accelerated rate of change in the inverse density (the specific volume) of matter in space over time. This is seen by dimensional analysis of the units of G: $(L^3/M)/T^2$, or ((volume / mass) / time) / time, a rather complicated triple ratio. If G is a constant then amount of space a unit of mass occupies is increasing (since G is positive) in an accelerated manner.

Ratios are routinely operated on as fractions and reduced to the lowest common denominator or replaced with the real number equivalent of the fraction. This discards information originally available, namely the sum or difference of the two original quantities. In the gambler example above, if the ratio of wins to losses (6:2) is reduced to 3:1 the sum does not reflect the number of games played. Similarly, with the ratio of wins to games played (6:8) reduced to 3:4 one can no longer find the number of losses.

Also, if 3:4 is expressed as a decimal, 0.75, again, almost all information is lost concerning the original event.

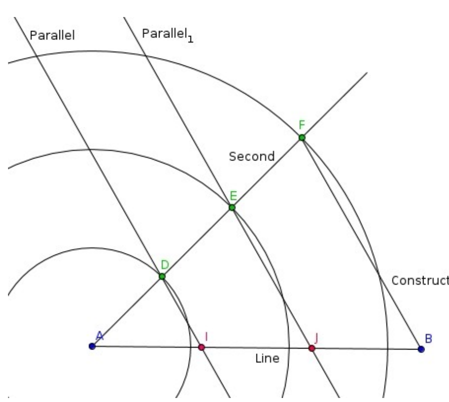
Proportions can be purely numerical: 3:4 :: 9:12 and 2:3 :: x:12. Proportions can also be of numbers and labels: 3:4 :: feet:seconds or equivalently feet:seconds :: 3:4. Rarely are proportions seen as purely label ratios: quarts:gallons :: quarters:dollars or strikes:outs :: feet:yards.

Not generally recognized, labeled proportion values are uncompleted operations. Velocity, as in 2 feet/1 second (an uncompleted operation) can be expressed as proportions: 2:1 :: ft:sec. Further, physical measurements yielding, for example, 10 feet in 5 seconds expressed as 2 ft/sec, should be written (and understood) as 10:5 :: ft:sec, a ratio formula, which is a proportion. Numerical ratios can be reduced by dividing both numbers by the second number when viewed as fractions, but this is just forcing the denominator to the value of '1'.

The formalization of uncompleted operations involving dimensions (labels) as proportions allows us to see the consequences of our choices of fundamental dimensions. In the MKS system (Meters, Kilograms, Seconds) there is no dimension of velocity, only a ratio of length to time (Meters/Seconds). We could adopt a VKS system (Velocity, Kilogram, Second) and in such a system there is no dimension of length. If an object under study were found to be moving (relatively of course) with a velocity of 2 vels (a contrived 'unit of velocity') for 5 sec it would travel a 'distance' of 10 vel sec.

Repeating Rational Numbers and the Irrationals:

The figure below shows a simple geometrical construction; a way to divide a line into any number of equal parts. Place two points, A and B, on the plane and draw the line between them. The object is to divide this line into x equal parts, in this example into 3 parts. Now place a point somewhere above the line towards the left end (point D below). Draw another line from point A through the point and about as long as the first line; the angle and length of this second line do not matter. Now using a compass set to the distance between A and D, mark off 3 equal segments on the second line. The actual lengths of the segments don't matter, only that they are equal in length. An important point here is that the compass arcs and the second line *are known to intersect,* as the distance between A and D can be considered to be 1, and the second line was constructed to intersect both. Points A, D, E, and F are all points placed on the second line with equal distances between them.



Now construct the line between points F and B, labeled ‘Construct’. Next draw lines through points D and E that are parallel to line Construct. By similar triangles we can prove that where these two lines (Parallel and Parallel_1) ‘cross’ the original line (Line) they divide that line into thirds. These points are labeled I and J. However, we are assuming that the parallel lines *intersect* the original line, that is, that the lines labeled Parallel and Parallel_1 have a point in common with the line labeled Line. *But intersection is a rather subtle concept.*

The geometrical definition of ‘intersection’ is an object such as a point, line, curve, etc. that is common to two or more objects. Two distinct straight lines that are not parallel will apparently intersect (on a Euclidean plane) but does that mean there is a point in common with both lines?

Rational numbers can be divided into two groups: those whose decimal representations have repeated zeros indefinitely on the right, and those whose decimal representations have a repeated group of one or more (but a finite number of) digits indefinitely to the right. The rational number ½ is of the first group: 0.5000… while the rational number 1/3 is of the second group: 0.333… . But this division into the two groups depends heavily upon the integer base of the number system in which you are representing the numbers. The preceding example is naturally in base ten. In base three the ‘decimal’ (trimal?) representations for ½ would be 0.111… and for 1/3 0.1000… . One can always find a base in which any rational number representation can be expressed with repeated zeros on the right. But no finite base allows for *all* rational numbers to end with repeated zeros. Irrational numbers are those for which *no* base allows for the representation to end with zeros.

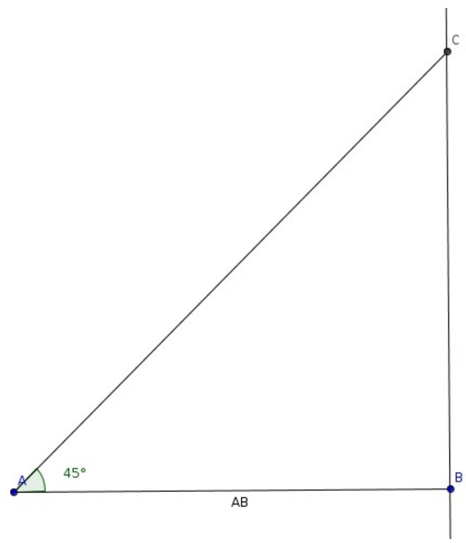
Rational numbers ending with repeated zeros can be placed on the number line precisely, the rational numbers that end with repeated groups of digits cannot be placed exactly on the number line, indeed, such points do not exist except as approximations. In attempting to place such a number on the number line one would have to sub divide the line infinitely many times. The point 0.333… would appear on the number line between point 0.333 and point 0.334, and between point 0.33333333 and point 0.33333334, but the actual point cannot be placed on the number line in a finite number of operations.

As described above, finding a number like 1/3 ($0.\overline{3}$) by construction, that is, dividing the number line between 0 and 1 into thirds, is problematic because it hinges on there being a point where two lines intersect. Despite thousands of years of Euclidean paradigm there is really no convincing argument that two distinct lines that are not parallel actually ‘intersect at a point’. The intersection in this example would only happen if 1/3 could be placed on the line, a circular argument.

However, if we were to do the above construction in base 3 the points I and J would be at 0.1 and 0.2, which can be placed on the number line exactly. But we (and the Greeks) do constructions with straight edge and compass, not with numbers in any base, so what does it mean to ‘do it in base 3’?

We conclude that the points that divide a line into three equal parts, 1/3 and 2/3 (or any rational number), do actually exist exactly and that the parallel lines and the original line do intersect at those points. The confusion arises from the attempts to give a numerical length to those segments in an inappropriate base number system.

So why go through this involved example? Because, as noted in passing above, irrational numbers are those for which *no* base allows for the representation to end with zeros, and therefore cannot be placed exactly on a number line, either by construction or by calculation. Consider the following diagram. We start with two points, A and B, a *measured distance* of 1 unit apart and construct the line joining them. (This is not a classical construction since we are measuring lengths.) Next draw a line from A at an angle of 45 degrees to the line $\overline{AB}$. Finally, construct a line through point B and perpendicular to line $\overline{AB}$. The question, both mathematical and philosophical, is does the 45 degree line intersect the perpendicular line?



We know that the length of line $\overline{AC}$ is the length of line $\overline{AB}$ times √2. We know the length of $\overline{AB}$ = the length of $\overline{BC}$. We know that √2 is irrational and can not be placed on a number line, in any base. This is, of course, Pythagoras' dilemma, brought from pure geometry into number theory.

If the point √2 does not exist on the line $\overline{AC}$ then there can be no intersection (point in common) with line $\overline{BC}$. This situation exists in the infinite continuous world of standard mathematics. In a finite and discrete math the situation is even more pronounced.

Imponderables:

There are many questions in mathematics that border on the philosophical, questions that deal with *definitions that make logical sense mathematically* but make little or no sense in the real world. Each of these concepts when first introduced met with severe criticism or even ridicule. It seems that many of these problems are the direct result of accepting uncompleted operations as numbers.

It must be strongly stated that given the inclusion of uncompleted operations in our mathematics these imponderable concepts are required to be defined as they are. They stretch credulity but keep mathematics reasonably consistent and allow for equation theory to continue to function. These concepts are not 'wrong' but they are cumbersome and hide much of the real world.

Many of these concerns are discussed elsewhere within this paper. Here are a few examples with their associated uncompleted operations:

Zero: from uncompleted subtraction of a number from itself.

Zero is the same as all the other integers except – it's different, very different. All integers are either positive or negative, except zero. All integers (at least the positive integers) express a quantity, except zero. How much do zero elephants weigh? Explain why any number times itself zero times is 1 ($13^0 = 1$). Why is 0! = 1, and does it make sense that 0! = 1!? Three factorial is 3 x 2 x 1 = 6 (unless you're a programmer looking to save code and time, then it's just 3 x 2 = 6). And of course, there is the 'small' problem of dividing by zero.

Negative one (and all the rest of the negative numbers): from uncompleted subtraction.

The positive numbers times -1 yields the negative numbers. Negative quantities do not exist. You can not find -2 elephants. (No, you can't 'owe' 2 elephants – that's just a different way of defining the symbol '-1'.) What is negative acceleration? An opposite direction? (Again, just a new definition of the negative sign.) An opposite direction should have its own label: south, not negative north. How is it that $13^{-1} = 1/13$? Thirteen times itself negative one times means what, exactly? How is it that the natural log of -1 is πi? What number times itself equals -1?

Rational numbers: from uncompleted division.

Non-integer rational numbers (those with something other than 1 in the denominator when reduced to lowest terms) can not express a quantity – certainly not when dealing with discrete objects (quantum valuations). At best a rational number is a re-scaling: ¾ of a pie is 'really' 3 times ¼ piece of pie. Pies can be subdivided, but what is ¾ of a gold atom? Between any two adjacent integers there are no integers – between any two rational numbers there are an infinite number of rational numbers (not to mention that there are no adjacent rational numbers!). How can the infinite sum of the rational numbers that comprise the digits of π be irrational?

Imaginary numbers: from an uncompleted square root of an uncompleted subtraction.

The product of two positive numbers is positive and the product of two negative numbers is positive, so what number when squared yields a negative one? Since there is no such number we pretend that there is and call it '*i*', for imaginary, and hope it 'goes away' before we reach a solution to a real computation. Like most other imponderables '*i*' requires its own special algebra.

Infinity and Cantor's fantasies: uncompleted additions (Σ to ∞), uncompleted multiplications (Π to ∞), uncompleted divisions by uncompleted subtractions (1/0).

Of course, mathematicians disavow infinity as a number. But we use it as if it were. Aristotle proclaimed two infinities: potential infinity – some set of things of unlimited number, and actual infinity – some collection of objects or ideas that is complete and not finite in number. Actual infinity has never been found to exist in the universe of things, and arguably not in the universe of the mind either. Potential infinity is just another way of defining 'without limit', which in practice always means finite in any particular instance. The possible exception is the concept of convergence 'at infinity' or 'in the limit', but even that seems to be just a workaround for the unobtainable infinite. Does the Mandelbrot fractal exist in its entirety, or is it simply unlimited in its detail?

Irrational and non-algebraic numbers: numbers by definition, like *i*, that can not be stated explicitly in a finite number of expressions.

Calculus requires a continuous set of real numbers, which requires irrational and non-algebraic numbers to be complete. Cantor's proof of the uncountability of real numbers while the rational number are countable (meaning there are the same 'number' of integers as rationals but that there are more irrationals than either) is based on the existence of actual infinities. *No one has ever used an irrational number in any calculation.* Only rational number approximations of irrational numbers are useful.

Continuous real number line: Positive integers arranged equally spaced on a line, plus all the uncompleted operation numbers except the imaginaries and transfinites.

All of the irrational 'points' on the real number line are found by defining a point (though not explicitly defining its position on the line), determining if it is rational, and if not, it must be irrational. Like many other definitions of convenience it assumes the result. There is an alternative to 'if it isn't rational it must be irrational', and it is 'if it isn't rational it doesn't exit'.

Calculus, integration and differentiation: use of infinitely large, infinitely small, and division by zero.

Yes, it works, and is an extremely valuable mathematical tool used in physics, engineering, and just about everywhere. But it is based upon imponderable questions too numerous for this short description.

Calculations:

The jokes about engineers and mathematicians are not as numerous as those of lawyers, blonds, or, from the much less PC 1960s, ethnic, racial, or other minority groups. But taking degrees in engineering computer science, and mathematics in the 60s and 70s I was exposed to the humor concerning the differences in the world views of engineers and mathematicians. While I started questioning some of the precepts of math even in high school (my ninth grade algebra teacher called me a 'rabble-rouser'), I saw no such difference in world views until much later. It seemed to me that engineers used the same math as mathematicians, they just rounded off answers to whatever precision was needed. Now I realize that the difference is quite real.

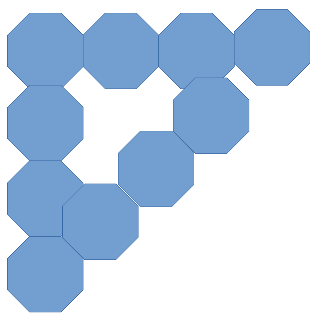
"Measure it with a micrometer, mark it with a crayon, cut it with an ax." This was a common remark concerning highway construction when I worked as a surveyor. We tried to hold all markers to within a hundredth of a foot, but watched as the bulldozers cut swathes to within a foot or two. Engineering was all about using math to whatever precision was required to get the job done. Often times small mistakes in measuring would be completely lost in the much rougher process of construction. Another saying from house construction summed it up, "hammer to fit, paint to match".

"An engineer and a mathematician stand at the door of a room. Across the room is a bed where a gorgeous naked lady beckons (remember, this is the 60s, almost all engineers and mathematicians are nerdy men). They are told they can have their way with her if they go to her by going first half way across the room, then going half the remaining distance, then going … etc. The mathematician gives up immediately saying it is impossible to reach her, while the engineer walks right over to her saying, 'close enough to count'."

This joke demonstrates that the engineer is not just rounding off to a set precision, he (and now she) actually sees reality differently. The engineer knows an isosceles right triangle can be built, the mathematician knows it can *not* be exactly designed. The engineer does not care that attempting to measure the legs and hypotenuse of the actual triangle using the same unit scale will not yield exact results. The engineer inhabits a macro world that is finite and discrete.

The mathematician invents a world that is continuous with unlimited precision, whether or not it reflects the actual world. This is epitomized by the real number line. Mathematicians shy away from discrete and finite objects, because they know such things often 'don't work out'. The world of the physicist dealing with plank time and plank distances and the world of the present day nanoengineer dealing with atoms *are* discrete and finite. They must deal with building blocks that do not allow just any configuration, as indicated by the discrete construction below demonstrating the incommensurable lengths of an isosceles right triangle, which must make use of the irrational number square root of two, a length that does not exist in a discrete and finite world. *Today's engineers must be aware of finite mathematics*, which, unfortunately, does not exist in a usable form today.

So what must we do to build an isosceles right triangle using atoms? The answer is a choice between two options: either we don't build it or we engineer it to some approximation. That is, to build it we must make the angles or the lengths a little bit 'off'.



A discrete construction with uniform building blocks.

Many children's construction toys are discrete and finite. Tinker Toys, Lincoln Logs, Erector Sets, and Legos, to name a few that I have enjoyed, by their discrete nature all restrict what can be built. Tinker Toys *can* build an isosceles right triangle because they include connecting sticks that approximate a length of the square root of two relative to the other sticks in the set. This means that the relative lengths of many of the connecting sticks in a Tinker Toy set are, in theory but not practice, incommensurate.

The macro world has been very successfully described by continuous infinite math and physics. This is because the macro world is composed of huge numbers of discrete building blocks. An isosceles right triangle with legs one meter long has an hypotenuse of 1.414213562… meters. On an atomic scale those legs are 1,000,000,000 nanometers long and the hypotenuse is 1,414,213,562. … nanometers long. Who is going to notice, out of almost one and a half billion units, that we are off by a few tenths of one unit?

But this success is due to scale not mathematical validity. To be rigorous in the real world mathematics must use finite, discrete, indivisible data and calculate solutions that are finite and discrete. If the solutions are not finite and discrete then we must make the choice indicated above:

admit there is in actual practice no solution, or admit that any result is a discrete approximation to the design specifications.

Calculations that arrive at a non-discrete solution require re-calculation to attempt to realize a solution in practice. In the isosceles right triangle with one meter legs, calculating the length of the hypotenuse yields $\sqrt{2}$ meters. Realizing that this is not a discrete result we first check to see if the required precision is obtainable without reaching the limit of the scale of the discrete building blocks of the material used. Tinker Toys and houses do not pose a problem. However, the discrete diagram pictured above has a calculated hypotenuse length of $3\sqrt{2}$ discrete units or about 4.2. (The actual length of the legs is 3 units, not four. The triangle is formed from the centers of the building blocks. If this is bothersome, consider the construction as using four atoms for one leg, three atoms for the second leg at right angles, and three or four (neither work) atoms for the hypotenuse.) We can not use a change in the precision of scale as a finesse to arrive at a discrete solution. Here we must say the hypotenuse is 5 units long but the angles are not as desired, or that the construction doesn't close, or some other indication of the impossibility of realizing the *non-discrete mathematical* design.

Any calculation done in standard mathematics whose result includes an uncompleted operation must be re-calculated to discover why the uncompleted operation wasn't removed from the equations, or whether the scale allows an approximation, or, perhaps, the admission that it can't be done.

<u>Arguments for Finiteness and Discreteness – the Slinky and the Wheel:</u>

Everything we have discovered in the universe is finite in number and extent. The only possibly infinite quantities or measures are those held within human imagination. While we think there are some things that might be infinite or infinitely divisible, we have yet to actually find such a demonstrable object. Long thought to be infinite and infinitely divisible are time and space. Philosophers have long questioned, and physicists now have indications, that this may not be the case. Indeed, there is the possibility that neither actually exist at all (That's space and time may not exist, not philosophers and physicists, although …). Instead, some property such as relative velocity (at the macroscopic scale, and some strange quantum at the Plank scale) may be the foundation of physics, and time and space just inventions of our minds. The only 'things' that are infinite in any manner at all are those constructs of our minds such as the real number line and numbers,.

Given that this is the case, it seems extraordinary that we have devised a mathematics in support of the physical sciences that is rife with infinitudes. The overwhelming evidence of discreteness and finiteness in the known universe is the single strongest argument for a discrete and finite mathematics. There are seemingly unlimited (but not infinite!) examples of this, in fact every physical property known is an example. Any counter examples are easily dismissed by asking, 'What material are you going to use to build it?'. An answer of anything that is made of matter or energy definitely has both discreteness and finiteness. An answer of 'you don't need to build it' means you must imagine it. For instance, the circle is divided by a protractor into degrees. We can subdivide the degrees indefinitely. So, the argument goes, build an ever larger protractor and there is an infinite subdivision available. Perhaps, in our imaginations, but only there.

Even before Zeno, philosophers wrestled with these issues. How does an arrow move? How is an infinite set of distances traversed in finite time? The concepts of space and time are ancient. There is a branch of philosophy devoted to space and time, and ontological arguments and metaphysical concepts are fascinating, but none of these endeavors have ever given us testable hypotheses. Ancient civilizations for which we have any surviving records all concerned themselves with quantifying space and time. Obvious relics such as Stonehenge give mute testimony to the efforts these societies expended on the issues. Interestingly, some societies had differing views from the classical descriptions, a few of which seem to be precursor to relativity: the Incas apparently considered space and time to be one conceptual object. More recently, Newton gave us a metric for 'doing physics', Einstein finessed the concepts to explain new discoveries in physics (and explain inconsistencies he foresaw), but now an ever increasing number of physicists doubt that time exists, and perhaps space does not either.

In metaphysical camps, to simplify somewhat, we have the realists that think space and time exist 'in reality', the idealists who think there is no reality beyond the human mind, and the anti-realists who think <u>things</u> 'really' exit but space and time do not: they are imaginations of the mind. Such philosophical arguments have concerned many mathematicians and much of math is imbued with their conclusions. Descartes famously said, 'Cogito ergo sum': 'I think therefore I am'. How much of his metaphysics is embedded in the Cartesian plane?

As observers, humans are limited and biased in many ways. We have but few senses each of which span a small portion of their spectrum. Even more telling yet more subtle, we have built in characteristics that overshadow everything we do. One such characteristic is the sense of time. We have heart beats, breath rates, walking speeds, and eating, sleeping, and other bodily functions that give us a sense of time. Dispensing with this built in clock, even in imagination, is very difficult, but that is what we need to do for the next mental experiment.

Imagine a spoked wheel on a track, and for concreteness imagine the wheel to be one meter in circumference and the track ten meters long. Also image a white disk with a red dot near its perimeter above the track. We can see the disk is revolving because the red dot moves. Here is the hard part: image we have no internal sense of time, so we can not tell by watching the disk if its motion is uniform. For example, we can not tell if it takes the same amount of 'time' to make each revolution. We can, however, count the revolutions of the disk, and we notice that the wheel goes from one end of the track to the other while the disk rotates twice.

Assuming counting, measuring distance, and rotations are valid concepts (possibly a dubious assumption) we can say the wheel travels 10 meters in 2 disk rotations, or an average of 5 m/r (meters per rotation). The really important point here is that we assume the disk rotates uniformly. If the disk were slowing down with time, speeding up with time, or randomly varying its speed of rotation with time we would not and could not ever know. The concept of 'with time' doesn't exist in this experiment, nor does 'speed of rotation'. *All of time keeping is just the noting of the ratio of two different discrete motions.* A clock does not 'keep time', it moves pointers around a disk in a manner we cannot prove is uniform. The cesium 'clocks' at the various bureaus of standards don't keep time or tell time. The cesium-133 atom oscillates 'very precisely', losing less than 1 second in 100 million years. How do we know it is a consistent and precise oscillation? We don't. But we assign 9,192,631,770 oscillations as the definition of one second. Atomic clocks count ticks in the form of oscillations, and oscillations are just motion. The standards offices just count oscillations, hoping they are uniform and that, if they are not, that it *doesn't matter*. It is quite possible that the entire universe 'slows down and/or speeds up' and that this behavior is undetectable; physical processes may be invariant with such conditions. If in our thought experiment the disk and wheel were both to slow down to half speed part way through the process we would still obtain an average speed of 5 m/r, and never know the difference. (Yes, invariance and synchronicity are quite a bit more complicated than this example demonstrates.)

Now enlarge upon our thought experiment adding a second wheel and track just like the first. Set the wheels in motion and note that the first wheel, as before, reaches the end of the track in two revolutions of the disk, but the second wheel is only halfway down the track after 2 revolutions of the disk. While we don't know the absolute speed of the disk or either wheel, we can state that the relative speeds of the two wheels is 2:1, or that the first wheel is twice as fast as the second wheel. (Again, this ignores the simultaneity issues of relativity.) Here we don't care if the disk keeps consistent, precise, uniform 'time', it doesn't matter in the calculations of the relative speeds of the two wheels.

So if everyone uses the same clock then we can pretend that time exists, but what we are really doing is comparing the motion of an experimental object to the motion of a cesium atom's motion. If time doesn't exist (but assuming for now that space does) then we can only measure relative displacements, that is, velocity of one object to the velocity of another. The situation becomes very complicated, however, because apparently we cannot all use the same clock.

Turning now to a different thought experiment, Richard James, in 1942, invented a toy that has been used for physics demonstrations (wave mechanics) in schools and universities ever since: the Slinky. We can use the Slinky as a discrete motion demonstrator. Image a Slinky at the top of a flight of stairs just starting down the steps. At first it is entirely on the top step, then as it first touches the next step down it is on two steps, next it leaves the top step and is entirely on the second step down, finally, its inertia flips the top over and it touches the next step down occupying two steps. The point is that its motion is entirely discrete, it is always, precisely, on exactly one or two steps. It is never on three steps, nor is it ever on no steps at all. Like a person crossing a stream on stepping stones, providing you don't fall or jump, you are always on exactly one or two stones. This is discrete motion (bipedal motion, to be sure).

If space exists and has a discrete (quantum) nature then motion must be something like the Slinky's action. If space doesn't exist then some property like velocity may be discrete and allow Slinky-like motion. If space exists and is continuous then we have no answers for Zeno.

Positive Axis Coordinate Systems:

One of Descartes' major contributions to mathematics was the fusion of geometry and algebra. With his plane with two perpendicular axes he allows us to visualize equations and formulas, and to make quantitative such geometric concepts as triangles and trapezoids. Of course, the Cartesian plane allows for much else besides. Here we will examine the normal Cartesian plane with two axes marked with real numbers, then a plane with four axes (rays) using only positive numbers, and the algebras that are inherent but hidden that deal with both. Next, we will look at a positive axes plane system with three axes and its algebra. And finally, we examine the complex plane and a 'rotational' algebra that mimics the complex plain in a three positive axes system.

Using as examples the Mandelbrot set and Gaussian integers, we come to an understanding of the *difference between number and coordinate,* which is the long awaited point of this section.

Almost all of these considerations apply to three space (and higher) as well as to the two space examples given here. The exceptions may be quaternions and octonions expressed on a positive axes system. I have not investigated dimensions higher than three and don't know if 'rotations' can be simulated in those systems.

We learn in early algebra that a coordinate is 'an ordered pair of numbers'. It is specifically *two numbers* whose order matters. The numbers represent positions on the two axes. The two axes are independent and span the space. 'Span' means that every point on the plane has a coordinate pair and that each point has only one coordinate pair.

Limiting ourselves to just the x axis for now, if we start at a point 3 units to the right of the origin and move 2 units in the positive x direction and then 4 units in the negative x direction we end up at a point 1 unit to the right of the origin. These moves can be verbalized in two ways: a positive number of units in one of two directions, or a signed number of units in a single direction. Thus the '4 units in the negative x direction' could be stated 'negative 4 units in the (positive) x direction'. In either case, to determine the final position one adds positive units in a positive direction, adds negative units in a negative direction, subtracts positive units in a negative direction and subtracts negative units in a positive direction. Thus, there is an inherent algebra in Cartesian coordinate computations. This is even more apparent in vector addition, which we will not discuss here.

Using the compass points to label the axes (as done in Section 2, Every Calculation is a Positive Integer Process) we now can form a set of four rays (E, N, W, and S, in that order) that have only positive numbers marking positions. A point might now have a coordinate (on a plane, not the earth) defined as an ordered set of four numbers, however, as a consequence of the underlying algebra at most two will be non-zero. The algebra is an extension of that discussed in Section 2: the pairs N and S, and E and W always have at least one value of zero, found by taking the absolute value of the difference and labeling it with the label of the largest. For concreteness we let the positive x direction be east, the positive y direction be north, the negative x direction be west, and the negative y direction be south. Thus, the coordinate (1,2,3,4) is the same as the coordinate (0,0,2,2). That is walking from the origin 1 step east, 2 steps north, 3 steps west, and 4 steps south puts you 2 steps west and 2 steps south of the origin.

The underlying algebra for the ENWS system is again a pairwise cancellation: E and W, and N and S normalize to the absolute difference labeled with the larger number's label. This means when adding east to east you just add but adding east to west you take the absolute difference and label it with the larger of the two directions. This is precisely the same algebra as used in the standard Cartesian plane where the sign of the coordinates dictates addition or subtraction. The four axes, ENWS, span the space and are *independent with positive coordinate algebra*.

The 'ordered' in 'ordered pair' and 'ordered four' just means we don't have to label the numbers. In the standard Cartesian plane we could express (2, -3) as -3y, 2x, and in the ENWS system (1,2,3,4) after normalizing to (0,0,2,2) could be expressed as 2s, 2w (or 2w, 2s). With labels both systems require at most two numbers to express the position of a point. A point on an axis means the other axis position is zero so (1,0) could be written as 1x, and (0,0,0,2) could be written as 2s. The origin presents a problem (as zero always does): (0,0) could be written as . (Yes, that was a space before the period.) It could also be written as 0x, or 0y, or -0x, or -0y, or 0x, -0y, etc.

If we are using positive axes (rays) systems we are free to use any number of axes that are independent and that span the space. In the plane this allows for two systems: the four ray system above (ENWS) and a three ray system that I've labeled in other papers as Red, Green, and Blue, spaced symmetrically (120 degrees apart) with the Red axis aligned with the positive x axis of the Cartesian system and the Green axis next in a counter clockwise direction. The three axes are independent in that no axis is the *positive* sum of the other axes, and they span the space such that every point has a coordinate. We must introduce the concept of normalization of coordinates alluded to above. In the three positive axes system the point (1,0,0) can also be described as the point (2,1,1) and (5,4,4) as well as an unlimited number of other coordinates. These coordinates of a point form an equivalence class any member of which can be normalized to the representative of the class by subtracting the smallest number of the three from each of the three numbers. Thus every unique representative describes exactly one point in the plane and each point has exactly one unique representative coordinate. These representative coordinates always have at most two non-zero numbers.

Normalization is just a form of the pairwise cancellation algebra described in the ENWS system. In subtracting the smallest number from each of the three numbers making up a coordinate we are recognizing that a 'negative number of units along an axis' is the equivalent of the sum of the same number of positive units along the other two axes. Thus -2R (2 units in the 'negative' Red direction) is equivalent to 2G + 2B. So, if we have a coordinate of (2,3,4) we see that 2 of the Green plus 2 of the Blue is equivalent to -2 of the Red, therefore adding -2R (that is, subtracting 2R) and subtracting 2G + 2B leaves the position of the point unchanged and results in a normalized coordinate of (0,1,2).

We now move on to examining the complex plane and its hidden problems. The standard complex plane looks just like the Cartesian plane except that instead of two real number lines at right angles for axes it has a real number line (the x axis) and an imaginary number line (the *i* axis, in place of the y axis). The axes are independent and span the space. However, now each point has an associated coordinate *and an associated number*, namely a complex number. Thus, the point (1,2) is now written as (1, 2*i*) as a coordinate and written as 1+2*i* as a number. Using the algebra defined for complex numbers, 'points' (coordinates) can be added and multiplied. Addition works just as it does in the Cartesian plane (for vectors), but multiplication (not defined in the Cartesian plane) induces a 'rotation' as well as a scale change. (Nothing actually 'rotates', the algebra of multiplication of complex numbers places the resulting point on the plane as if rotated, but, again, nothing is rotated.) These are artifacts of complex algebra, but these characteristics find many applications in physics and electronics. *The rotation turns out to be useful but its association with the square root of negative one has no particular relevance.* This can be seen by forming a corresponding algebra in a three positive axes system.

Given a RGB positive axes system we can impose an algebra defined by the following rules:

(a,b,c) + (d,e,f) = (a+d, b+e, c+f)

(a,b,c) * (d,e,f) = (ad+bf+ce, ae+bd+cf, af+be+cd) or (ad+bf+ce, ae+bd+cf, af+be+cd)

where the ordered triplet (a,b,c) represents aRed, bGreen, cBlue.

In words, red does not rotate any color, green rotates counter clockwise 120 degrees, and blue rotates clockwise 120 degrees. (Once again, there is no actual rotation; a better description would be transformation.)

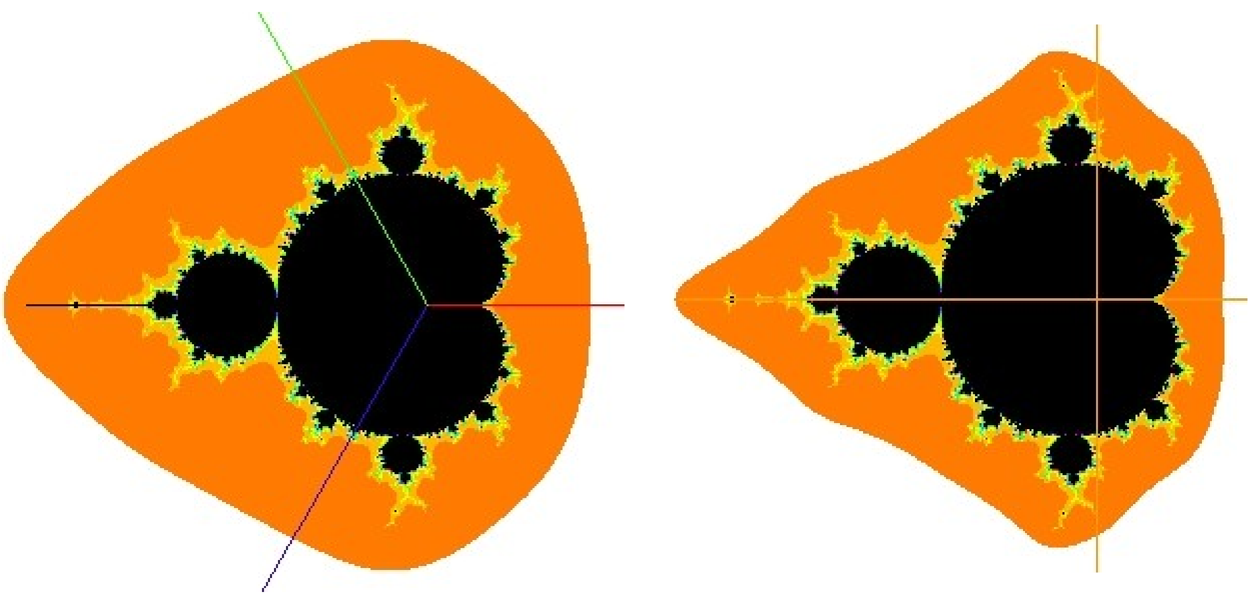
This is similar to the complex plane where positive x does not rotate, positive *i* rotates 90 degrees counter clockwise, negative x rotates 180 degrees, and negative *i* rotates 90 degrees clockwise. *Except that in the three positive axes system with rotation there is no association with imaginary numbers.* (As an aside, quaternions define three different types of imaginary numbers but here it is much easier to see that what is really being defined is a special algebra that addresses four types of numbered axes.)

The duality of both coordinate and number is rarely fully explored in math texts. A quick look at two examples expressed in both the complex plane and a positive three axes system with rotational algebra demonstrates why it should be.

The Mandelbrot set is an image recognized by mathematicians and non-mathematicians alike. It is both deep math and art. It takes Descartes' blending of geometry and algebra to an entirely new level that was all but impossible before the advent of computers. (Jules and Kock both created 'monsters' by hand.)

The Mandelbrot set iterates a formula on the *associated number* of the points on the complex plane. That is, it uses the coordinates of each point as a complex number.

The well known image is exactly duplicated using the symmetric three positive axes system with rotational algebra. Evan though the coordinates and the associated number of almost every point are both quite different the resulting computations based on the associated number yield the same outcome.

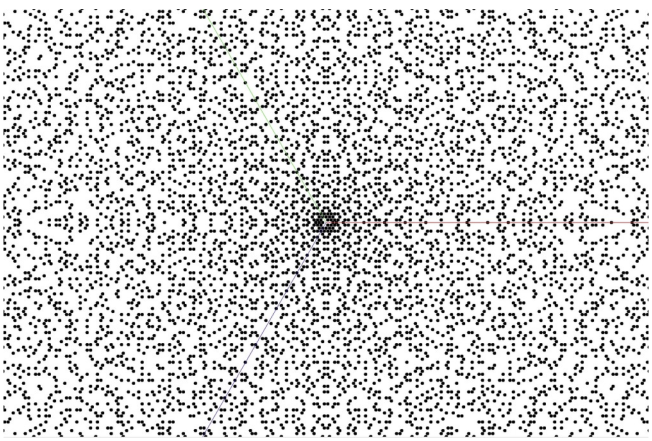


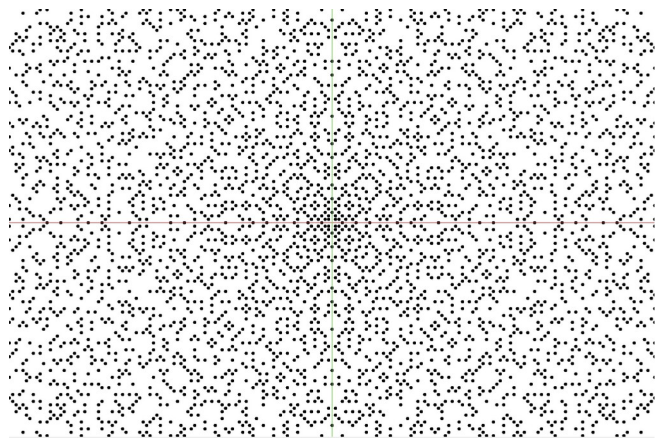
Mandelbrot generated with a positive three axes system on the left, and generated with the normal complex plane on the right. Both images from programs written by the author.

Non-symmetrical three positive axes systems yield skewed images of the standard Mandelbrot image. Only when the norm calculated by the product of the point and its conjugate equals the calculated squared distance between the point and the origin is the original image recreated. That happens only when the angle between the axes is 120 degrees.

Gaussian integers are those points on the complex plane whose *coordinates* are both integers. Various characteristics of this set of numbers derived from the coordinates include their being an integral domain. Such things as Gaussian primes, coordinates whose associated number is not a product of any two other Gaussian integers' associated numbers, can be found and mapped on the complex plane.

On a positive three axes system with rotational algebra Gaussian integers can be tested to see if they are prime or compounded of two other Gaussian integers. The resulting primes can be mapped on the positive three axes system and it is found that the mapping is entirely different from the complex plane Gaussian prime mapping. Other differences include zero divisors; investigation of this and other interesting anomalies is ongoing.





Gaussian Primes generated with a rotational positive three axes system on the left, and generated with the normal complex plane on the right. Both images from programs written by the author.

Set Theory Creates the Natural Numbers:

With apologies to set-theorists, I have little patience for set theory's attempt to create the natural numbers. It may be that I just don't understand set theory, but as my bias against zero is apparent in this paper and as set theory builds all the counting numbers from a form of zero, my distrust is, perhaps, understandable.

There are actually several different 'set theories' but by far the most common is the ZF (Zermelo-Fraenkel) set theory with its extension the ZFC theory which includes the axiom of choice. The ZF is an axiomatic set theory and defines the natural numbers this way:

{} first we have the 'empty set', where the curly braces indicate a set
{{}} then we have the set that contains the empty set

{{}{{}}} then the set that contains the empty set and the set that contains the empty set etc.

The empty set is often designated with the symbol ⊘ (not a Greek letter). We can equate the above sequence to the natural numbers (which for set-theorists includes zero) like this:

{} = ⊘ = 0
{{}} = {⊘} = 1
{{}{{}}} = {⊘ {⊘}} = 2
etc.

This is actually how the cardinality of sets is defined: as the number of elements in a set.

However, the empty set (no longer called the null set as that term has been usurped by other disciplines) has some very bazaar properties that recall similar problems with the 'number' zero. As an example of the problems with set theory that parallel the inclusion of zero into number theory, we have the set theory axiom that 'all sets include the empty set', except, of course, the empty set itself! Let's take a look at the definitions.

We start the investigation of ZF empty sets with these definitions: let A and B be sets and let $\mathscr{U}$ be the universal set which includes everything (another dubious definition). The empty set is as above: ⊘. The concept of a subset is defined such that A is a subset of B if and only if every element of A is also an element of B. This concept is written as A ⊆ B. Two sets are equal if and only if they contain the same elements.

There is a subtle point about subsets and the empty set. Subset A is formed by selecting elements of set B, and one can select all the elements, some of the elements, or *none of the elements*, the latter of which forms the empty set. The same dilemma arises here as with zero: is selecting no element the same as not selecting any element? If 'no element' *can be selected* from set B it must have been an element of set B; on the other hand, if set A is formed by making selections from set B then it can't be formed by *not making a selection*. Either way seems to be logically invalid.

Another point, the set ⊘ is defined to be unique, that is, there is only one set that is the empty set. Two sets are equal if they 'contain the same elements'. The assumption here is that since ⊘ has no elements no other set could have the same elements. This means that the set of zero elephants and the set of zero goats are identically the same set. *This is the same as saying zero is not a quantity*. If zero were a quantity, a number, then the two empty sets of elephants and goats would not be identically the same, they would be equivalent as defined as having the same number of elements, in this case zero, *but not equal*. I view this result, that zero is not a quantity, as a good thing but I do not particularly care for how this conclusion was obtained.

These considerations have a considerable history of famous mathematicians trying to resolve paradoxes such as Russell's, Skolem's, and Banach-Tarski's with new definitions and axioms such as the axiom of choice, the well-ordering theorem, and Zorn's lemma. Much of this work was the impetus for Godel's most notorious work: the incompleteness theorems.

I don't agree with any of the spider web constructions which are based on zero as a number or an actual infinity.

Discrete Derivative:

It is interesting to attempt a brute force method of finding a derivative of a function describing a discrete quantity. Using the example from Section 6, Infinity and Zero, we try letting the increment 'go to' 1 instead of zero. A discrete quantity should be measured in units of its smallest discrete size, and the '1' in the derivative formula is just 1 of those discrete units. Actually, the increment doesn't have to 'go to' anything, since there is no need to try to circumvent division by zero with infinitesimals or limits.

$f(x) = x^2$,

$f'(x) = ((x+h)^2 - x^2)/(x+h) - x$

$= (2xh + h^2)/h$

$= 2x + 1$, when setting $h = 1$

This result does not yield correct results – too much brute and not enough understanding. There is a branch of mathematics, finite difference calculus, that addresses this issue. However, a more refined method of defining both derivative and integral operations on discrete functions can be found in my paper "Quantum Calculus" included here in its entirety, which gives beautiful results:

The Difference Path to Bernoulli Numbers: Developing a Quantum Calculus

Calculus, also called infinitesimal calculus, is the analysis of continuous change. Discrete calculus, despite its name, is the analysis of continuous change approximated by Riemann sums of various intervals of the (continuous) independent variable. If the interval used in discrete calculus is allowed to approach zero discrete calculus becomes infinitesimal calculus.

Quantum calculus (my name for this effort) is similar to discrete calculus but its purpose and process are quite different. As science pushes into both smaller and larger realms of the universe it is becoming apparent that at least some (if not all) 'things' have a smallest amount. That is, things are made up of one or more discrete and indivisible bits. Even space and time may be as discrete as a bar of gold, which can only be divided into atoms of gold; beyond atoms it can not be divided and still retain 'goldness'. Quantum calculus does not approximate a continuous function, rather, it defines a calculus designed to quantify discrete functions and processes – specifically, quantify discrete processes measured in indivisible units. Quantum calculus defines the the *h* of infinitesimal calculus to be 1.

It is not too surprising to find that the Bernoulli numbers appear in the generation of quantum integration, but before we rediscover Bernoulli numbers we must define quantum differentiation.

The starting point of any calculus is rate of change of one measurable quantity against the change of another, called differentiation. Standard nomenclature has a function *f* applied to an independent variable *x* that yields a value of the dependent variable *y*, expressed as $y = f(x)$. As *x* changes so might *y* depending on the function *f*. A new function can perhaps be found that expresses the rate of this change, and finding this new function is the purpose of differential calculus. Infinitesimal calculus defines differentiation with this equation:

$$\frac{dy}{dx} = \lim h \rightarrow 0 \, \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

Quantum calculus simply replaces *h* with the number 1 and does away with any application of a limit. This yields the well known forward difference equation:

$$f(x+1) - f(x)$$

for finding the quantum differential of the function *f*. There are difficulties with indices using this formulation, however, so, instead of forward difference, quantum calculus uses backward difference which also gives an appropriate modified form of Pascal's triangle, useful later. The equation for quantum differentiation is then:

$$f(x) - f(x-1)$$

It is understood that the independent variable *x* can take on only integer values, since its smallest value is 1 and any larger values must be integer multiples of 1. The dependent variable is expected to also be integer valued.

In this paper only polynomial functions will be examined, so we start with finding the quantum differential equations of powers of *x*. If $f(x) = x$ then the backward difference equation yields just 1, hence the quantum derivative of *x* is 1. For $f(x) = x^2$ applying the backward difference formula yields the derivative 2x-1. In fact, we find the first several values of the derivatives of powers of x to be:

f(x)	derivative of f(x)
x	1
x^2	$2x-1$
x^3	$3x^2-3x+1$
x^4	$4x^3-6x^2+4x-1$
x^5	$5x^4-10x^3+10x^2-5x+1$
x^6	$6x^5-15x^4+20x^3-15x^2+6x-1$
x^7	$7x^6-21x^5+35x^4-35x^3+21x^2-7x+1$
x^8	$8x^7-28x^6+56x^5-70x^4+56x^3-28x^2+8x-1$
x^9	$9x^8-36x^7+84x^6-126x^5+126x^4-84x^3+36x^2-9x+1$
x^{10}	$10x^9-45x^8+120x^7-210x^6+252x^5-210x^4+120x^3-45x^2+10x-1$
x^{11}	$11x^{10}-55x^9+165x^8-330x^7+462x^6-462x^5+330x^4-165x^3+55x^2-11x+1$
x^{12}	$12x^{11}-66x^{10}+220x^9-495x^8+792x^7-924x^6+792x^5-495x^4+220x^3-66x^2+12x-1$
x^{13}	$13x^{12}-78x^{11}+286x^{10}-715x^9+1287x^8-1716x^7+1716x^6-1287x^5+715x^4-286x^3+78x^2-13x+1$

Table 1: Quantum Differentiation

Using the notation of ‘D’ for quantum differentiation (and later, ‘I’ for quantum integration) we also find that D is linear:

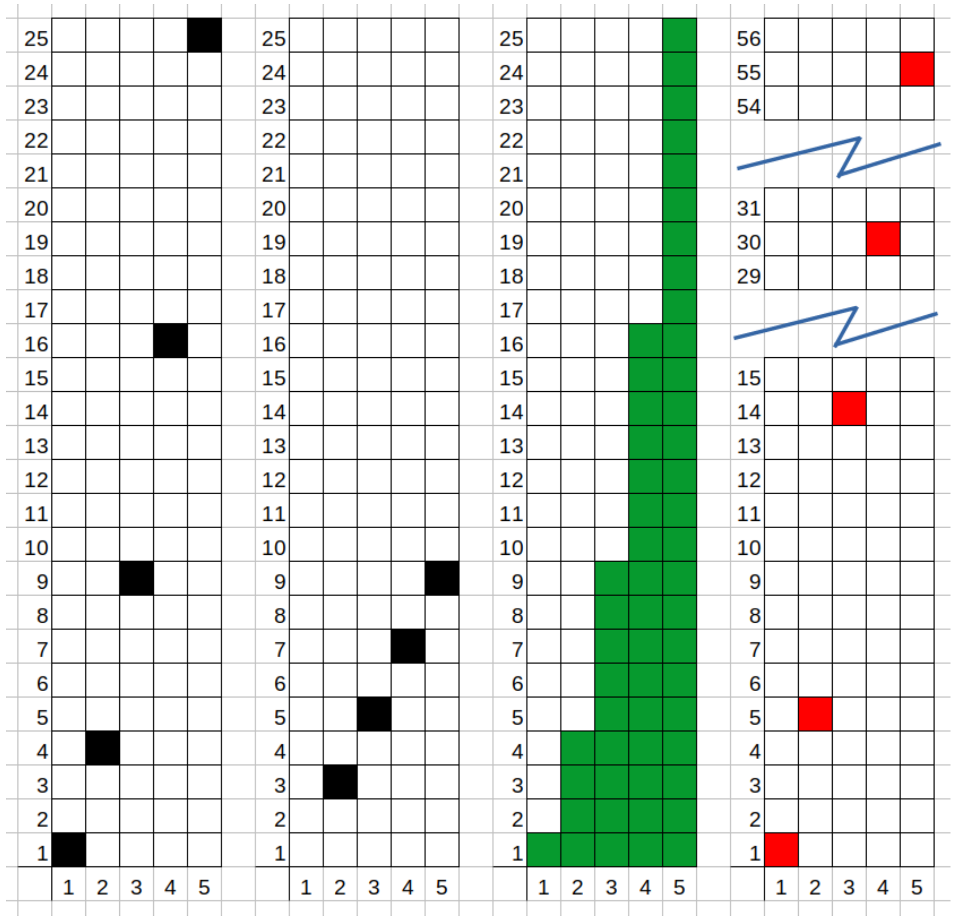
$$D(cx^n) = cD(x^n), \text{ c a constant coefficient, and}$$
$$D(x^n + x^m) = D(x^n) + D(x^m),$$

and other normal calculus identities hold as well.

Several characteristics of the above table of quantum differentiation are intriguing. The coefficients seem very close to those of Pascal’s triangle. Starting with Pascal’s triangle if we eliminate the leading 1 from each row (thus eliminating the first row altogether) and then negate every other term starting with the now second term of each row, we arrive at exactly the above table of coefficients. This seems rather remarkable, but is due to the backwards difference formula: it is simply a binary expansion of a difference less the leading term. If we sum the coefficients of any row of the above table we find they always sum to a value of 1; again, rather remarkable. And, if the coefficients of Table 1 are entered into a lower triangular square matrix the matrix is involutory, that is, it is its own inverse.

As opposed to continuous calculus which defines a slope at every point (at every real value of x for continuous differentiable functions) quantum calculus can only define dependent variable change between two adjacent independent variable values. There are no points on a line representing slope since the values represent discrete quantities and there are no intermediate values between indivisible units of measure. Consequently we can correctly speak only of the change in *y* with unit change in *x*, not the slope of the line between two values of *x*. When we solve for the rate of change at ‘point’ $x = 3$, for example, using the above formulas, we obtain the change in *y* from $x=2$ to $x=3$.

We now have two methods of finding quantum difference equations: the backwards difference formulation and the modified Pascal’s triangle formulation. With the quantum differentiation formulas in place we can move on to establishing the integration formulas for polynomials. Today there are several methods of developing the integration formulas, some rather esoteric. Historically, Bernoulli, Faulhaber, and others, were solving the sum of powers of integers problems, which led to the Bernoulli numbers and a general though rather complicated solution. An example of the type of problem posed is: what is the sum of the first 5 integers each raised to the 2nd power? **This, of course, is also the quest for the quantum integral of x^2 from 1 to 5.** Thus if we know the formula for the sum of cubed integers we also know the formula for the quantum integral of x^3, and the reverse is true as well. A diagram of values, rates of change, area under values, and sum of powers of integer concepts for quantum calculus makes this plain:



X, Y values $y=x^2$ | Rate of change $y=2x-1$ $D(x^2)$ | Areas | Σ (integers)^2 $1/3(x^3)+1/2(x^2)+1/6(x)$ $I(x^2)$

Table 2: Quantum Differentiation and Quantum Integration

It's tempting to draw lines between the squares of the first, second, and fourth plots but we cannot. A point on such a line would of necessity subdivide the smallest lengths labeled 1, 2, 3, 4, and 5. The Areas plot shows the equivalence of quantum area and sum of powers of integers. Notice that the coordinate system labels 'spaces' not 'lines' and that the axes start with 1, not 0. This is a quantum graph on quantum graph paper.

Several conventions have arisen around the Bernoulli numbers that must be explicitly stated and, perhaps, improved upon. From algebra we routinely express polynomials in descending order of degree, as in Table 1. This is pure convention and we can impose a modified Pascal's *triangle* on a *square* matrix in at least eight ways, as we will see. The actual Bernoulli numbers are sometimes put in table form omitting all zero values, sometimes retaining the zeros, sometimes the second number is written as a positive ½, and sometimes as a negative ½. When the expressions for the sums of integers to a certain power are written they sometimes include the zero coefficient terms, sometimes not, and are sometimes written in descending order of degree of term, sometimes not, changing how the terms are labeled (numbered). Further, the conventions of matrix manipulation are embedded in calculator procedures. Taking the inverse of a matrix, for example, depends on row calculations giving rise to seemingly non-symmetric results, demonstrated later.

In this paper the following conventions apply unless otherwise explicitly stated:

- Polynomials will be expressed in descending degree of terms
- Polynomial terms are numbered from the left starting with 1, regardless of order of degrees
- The Bernoulli numbers are labeled (numbered) from zero
- The Bernoulli numbers include the zeros (every odd labeled number except B_1)
- The second Bernoulli number (B_1: ½) will always be positive
- Inverse quantum differentiation matrices do include zero terms
- Quantum integration formulas do not include terms with zero coefficient
- Quantum integration formulas entered into a matrix do include zero terms

There follows a direct brute force method for finding quantum integration formulas which builds upon the already discovered quantum differentiation formulas and is in keeping with discovering the underpinnings of a quantum calculus. Other, better, methods are discussed later in this paper.

This technique for finding an integration formula for a power of the variable, x^n, relies on reiterative application of the previously determined differentiation formulas. Any candidate for the integral must conform to the composition of differentiation and integration functions yielding the original variable. Letting D(f(x)) be the quantum differentiation operator and I(f(x)) be the quantum integration operator we must ensure that:

$$D(I(f(x^n))) = f(x^n).$$

A simple example demonstrates the process, followed by a much longer example showing the intricacies giving rise to the zero coefficients of almost half the terms. First we find the integral of the variable x to the first power (dropping the ‘f()’ notation):

$I(x^1) = ?.$

$D(I(x^1))$ must equal x^1, so we try $I(x^1) =? x^2$.

From the quantum differentiation formulas $D(x^2) = 2x-1$, not x.

Next we try $I(x^1) =? 1/2(x^2)$,

but find $D(1/2(x^2)) = 1/2D(x^2) = x-1/2$, still not x.

To eliminate the -1/2

we add a term to the integral $I(x^1) =? 1/2x^2+1/2x$

and find $D(1/2x^2+1/2x) = x-1/2+1/2 = x$

so $I(x^1) = 1/2x^2+1/2x.$

Next we find the integral of x^8, using the following table:

	to find I(x^8)									1
								3	-3	1
						5	-10	10	-5	1
				7	-21	35	-35	21	-7	1
			8	-28	56	-70	56	-28	8	-1
		9	-36	84	-126	126	-84	36	-9	1
9	1/9	1	-4	9 1/3	-14	14	-9 1/3	4	-1	1/9
8	1/2		4	-14	28	-35	28	-14	4	- 1/2
			0	-4 2/3	14	-21	18 2/3	-10	3	- 7/18
7	2/3			4 2/3	-14	23 1/3	-23 1/3	14	-4 2/3	2/3
				0	0	2 1/3	-4 2/3	4	-1 2/3	5/18
5	- 7/15					-2 1/3	4 2/3	-4 2/3	2 1/3	- 7/15
						0	0	- 2/3	2/3	- 17/90
3	2/9							2/3	- 2/3	2/9
								0	0	1/30
1	- 1/30									- 1/30
		1	0	0	0	0	0	0	0	0

Table 3: Finding the Integral of x^8

Most of the arithmetic done in finding I(x^8) is done on the formula for D(x^9), so we enter that formula in the row between the bold lines. The aim is to add terms to the proposed integral that cancel out all the terms of the of the D(x^9) except the first term, leaving the first term with a coefficient of 1. The first term in the D(x^9) is degree 8 which is the goal of D(I(x^8)).

The coefficients of the derivative of x^9 are inserted in the cells between the bold lines. Then that row is multiplied by the inverse of the first coefficient with the result put in the next row down. To the left of that row is shown first the degree of that term *of the integral to be found*, and next the multiplier of that row, 9 and 1/9 respectively. We want to keep the first term, 1, representing $1x^8$,but wish to eliminate the second term, which is -4. To do that we enter in the row above the double bold line row the values of the derivative of x^8, then multiply that row by whatever it takes to eliminate the -4, in this case ½, and the values are entered into the next row and preceded by 8 indicating x^8 and ½ indicating the multiplier. This row is then added to the row above it with the results entered into the next row down. This process is continued until the last remaining unwanted term is eliminated. Notice that sometimes two terms are eliminated at the same time, which is why there are skipped degrees after the first three entries.

Finally, the answer is read from the two columns on the right, in this case:

$$(1/9)x^9+(1/2)x^8+(2/3)x^7-(7/15)x^5+(2/9)x^3-(1/30)x.$$

The integration table (Table 4, shown below) has several intriguing patterns, for some of which it is hard to quantify the underlying structure. The coefficient of the first term of an integration of any power function, x^n, is just 1/(n+1). The second coefficient is always ½. The third coefficient, starting with the integration of x^2, is a multiple of 1/12: n/12. The fourth term is always zero, as is every even term after that.

There are several additional fascinating properties of the table of discrete integration. First, despite the fractional coefficients an input of an integer value of x always yields an integer value for I(f(x)). And like the derivative table, the sum of the coefficients of any indefinite integral (excluding a constant of integration) of x^n is always one. Finally, the coefficient of the x^1 term of the discrete integration of x^n is Bernoulli number B_n. (Using the convention that $B_0=1$, $B_1=1/2$, $B_2=1/6$, and B3=0…).

f(x)	indefinite integral of f(x)
c	1/1(x)
x	$1/2(x^2)+1/2(x)$
x^2	$1/3(x^3)+1/2(x^2)+1/6(x)$
x^3	$1/4(x^4)+1/2(x^3)+1/4(x^2)$
x^4	$1/5(x^5)+1/2(x^4)+1/3(x^3)-1/30(x)$
x^5	$1/6(x^6)+1/2(x^5)+5/12(x^4)-1/12(x^2)$
x^6	$1/7(x^7)+1/2(x^6)+1/2(x^5)-1/6(x^3)+1/42(x)$
x^7	$1/8(x^8)+1/2(x^7)+7/12(x^6)-7/24(x^4)+1/12(x^2)$
x^8	$1/9(x^9)+1/2(x^8)+2/3(x^7)-7/15(x^5)+2/9(x^3)-1/30(x)$
x^9	$1/10(x^{10})+1/2(x^9)+3/4(x^8)-7/10(x^6)+1/2(x^4)-3/20(x^2)$
x^{10}	$1/11(x^{11})+1/2(x^{10})+5/6(x^9)-1(x^7)+1(x^5)-1/2(x^3)+5/66(x)$
x^{11}	$1/12(x^{12})+1/2(x^{11})+11/12(x^{10})-11/8(x^8)+11/6(x^6)-11/8(x^4)+5/12(x^2)$
x^{12}	$1/13(x^{13})+1/2(x^{12})+1(x^{11})-11/6(x^9)+22/7(x^7)-33/10(x^5)+5/3(x^3)-691/2730(x)$

Table 4: Quantum Integration

There are other ways to find the quantum integration formulas. Since they are exactly those of the ‘sum of powers of integers’ formulas the techniques for finding the sum of powers of integer formulas also finds the quantum integration formulas. There are generalized formulas for the sum of powers of integers that work for any power of the variable given a known table of Bernoulli numbers, and there are formulas for finding the Bernoulli numbers, as well. There are formulas that combine the two. These generalized formulas are somewhat cumbersome, involving falling factorials, factorials, the Bernoulli numbers, combinatorics, summations, and double summations of many terms.

Ignoring the ‘sum of powers of integers’ path altogether and looking only at the simply derived quantum differentiation formulas there is an absolutely fascinating way to find the quantum integration formulas, and so also the sum of powers of integers formulas as well as the Bernoulli numbers.

If the coefficients of Table 1: Quantum Differentiation are entered into a lower triangular matrix reversing the order of the terms of each row, and then the inverse is taken we obtain the matrix shown here:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{12} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{42} & 0 & \frac{-1}{6} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{12} & 0 & \frac{-7}{24} & 0 & \frac{7}{12} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{30} & 0 & \frac{2}{9} & 0 & \frac{-7}{15} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{9} & 0 & 0 & 0 & 0 \\
0 & \frac{-3}{20} & 0 & \frac{1}{2} & 0 & \frac{-7}{10} & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{10} & 0 & 0 & 0 \\
\frac{5}{66} & 0 & \frac{-1}{2} & 0 & 1 & 0 & -1 & 0 & \frac{5}{6} & \frac{1}{2} & \frac{1}{11} & 0 & 0 \\
0 & \frac{5}{12} & 0 & \frac{-11}{8} & 0 & \frac{11}{6} & 0 & \frac{-11}{8} & 0 & \frac{11}{12} & \frac{1}{2} & \frac{1}{12} & 0 \\
\frac{-691}{2730} & 0 & \frac{5}{3} & 0 & \frac{-33}{10} & 0 & \frac{22}{7} & 0 & \frac{-11}{6} & 0 & 1 & \frac{1}{2} & \frac{1}{13}
\end{pmatrix}$$

Table 5: Inverse of Quantum Differentiation Matrix

Notice two things: First, **each row is the quantum integral** for the corresponding power of x (row n is the formula for $I(x^{n-1})$), in order of ascending degree starting with degree 1.

And second: the **Bernoulli numbers are displayed** in the first column.

All of this with just the push of a calculator button to invert a matrix!

The differentiation formulas must be entered in reversed order (lowest degree term first). Investigating this requirement brings to light some additional interesting aspects of the quantum differentiation coefficient matrix.

Strangely, if the differentiation coefficients are entered highest degree first in a lower triangular matrix the matrix is involutory. That is, the matrix is its own inverse, the inverse yielding the differentiation matrix again, rather than the integration matrix.

Starting with that involutory matrix there are three things (at least) we can do to rearrange the entries. We can right or left justify the terms within the matrix, we can reverse the order of the terms, and we can enter the rows starting at the bottom of the matrix (upside down, the single term row at the bottom of the matrix). We can abbreviate these rearrangements right/left for justification, std/rev for standard (highest degree first) or reversed order, and nrm/inv for normal (single term formula in row 1) or inverted (single term formula in bottom row). This gives us (without using transposition) eight permutations. Due, apparently, to the matrix inversion algorithm there is no arrangement that yields a left/std/nrm matrix of integrals from a left/std/nrm matrix of differentials.

As noted, the inversion of the left/std/nrm differential matrix yields the lft/std/nrm differential matrix. There are two such involutory arrangements: left/std/nrm and right/rev/inv.

There are two cyclic involutory arrangements, that is, the inverse of one yields a rearrangement of the entries the inverse of which brings back the first arrangement: right/rev/nrm to left/std/inv.

There are four arrangements whose inverse does yield the integral coefficients;

left/rev/nrm to left/rev/nrm, right/std/nrm to left/rev/inv,
left/rev/inv to right/std/nrm, and left/std/inv to left/std/inv.

Of these the first and last are most useful – the first generated Table 5.

Also of note, if the matrix of quantum integration coefficients are entered in reversed order that matrix is not invertible.

In summary:

This paper demonstrates a discrete form of calculus, both differentiation and integration, that yields correct answers for any process that depends upon discrete indivisible units. Processes that are quantum-like may find this form useful, and unlike classical calculus discrete calculus has no ambiguities around infinitesimals, division by zero, or ignoring non-zero values.

The combinatorial formula for generating the quantum differential of the terms of a polynomial (not shown here) suffices for physical calculations, however, the general formulas for the sum of powers of integers (hence quantum integration) are cumbersome. The inverse matrix method of finding the quantum integration formulas (hence the sum of powers of integers formulas) is very simple.

There is the interesting equivalence that arises from these efforts. Letting the integral symbol with ‘Q’ in place of the lower limit and ‘n’ as the upper limit indicate a quantum integration evaluated at n, we have:

$$\sum P(n)=\int_{\mathrm{Q}} P(x)$$

That is, the sum of a power of the integers polynomial evaluated at integer values from 1 to n is equal to the quantum integral of that polynomial evaluated at n. For example:

$$\sum_{n=1}^{t}(2n^3+5n^2-n)=\int_{Q}^{t}(2x^3+5x^2-x)$$

which yields from the quantum integration tables:

$$\frac{1}{2}t^4+\frac{8}{3}t^3+\frac{5}{2}t^2+\frac{1}{3}t$$

also supplying the formula for the sum of powers of integers, as well.

Setting aside sum of powers of integers calculations and combinatorial rising factorial formulations, and looking only at quantum calculus formulas, we have the following straight forward set of conclusions:

1) the backwards difference formula yields the quantum differentiation formulas.
2) those formulas placed into an appropriate matrix form a modified Pascal’s triangle.
3) the inverse of that matrix yields the quantum integration formulas.
4) those formulas yield the Bernoulli numbers and sum of powers of integers formulas.

Since the backwards difference formula is a binomial expansion less the first term it is little wonder it yields a modified Pascal’s triangle. Since differentiation and integration are inverse functions it is small wonder that inverting the matrix would transform between the two. And since the sum of powers of integer equations are precisely the quantum integration equations it is no wonder the Bernoulli numbers are found in the inverse matrix of quantum differentiation.

It is known that the difference formulas (both forward and backward) yield modified Pascal’s triangles, and it is known that a suitably modified Pascal’s triangle in matrix form can be inverted to find both the sum of powers of integers formulas and Bernoulli’s numbers. However, that the quantum differentiation and quantum integration formulas can be expressed as inverse matrices of each other is not generally realized, nor is the connection between quantum integration and Bernoulli’s numbers.

Section 8: Invented Math and Future Directions

Discovered vs Invented Mathematics:

A long standing argument within the philosophy of mathematics arena concerns how much of mathematics is 'out there' and discovered by humankind, and how much of mathematics is 'in the mind of mathematicians' and therefore invented by humankind. If such a line must be drawn, it seems, considering the preceding arguments, that only the natural numbers, the counting positive integers, with addition and limited subtraction, have been discovered and all the rest of mathematics – all the various types of numbers and all the other operations are the invention of mathematicians.

Rebels and What Next:

Throughout most of the development of mathematics the heretics of math have been those who propose a new type of number (e. g. square root of negative one) or a new type of operation (calculus), and they have met much resistance and even ridicule. Things have changed.

Today mathematics has so many areas of specialization that nobody can take it all in. When a new approach to a problem involves a new type of number or operation it is just one of a huge number of such objects in the math environment, and is accepted by the community without much thought. It is quite acceptable to enlarge the field with more and more abstractions and uncompleted operations posing as numbers. The mavericks of mathematics today are those few who are demanding a more rigorous examination of what has been accepted as obvious and logical.

While the math beyond the natural numbers is extremely useful, it does suffer from the rather ill-considered view of the overall structure of mathematics by its practitioners. Without condemning the existing structure as wrong, some are decrying the lack of definition and rigor of mathematic's very foundations. One such mathematician is Norman, J. Wildberger, PhD. He has written several books and has a web site and many YouTube videos that are well done and educational, as well as presenting some radical ideas (actually, non-radical, as he denies the existence of irrational numbers). His work is well worth reviewing, but is seemingly ignored by the mainstream math community.

Wildberger concludes that all derived numbers beyond the rationals (integer fractions) are nonsense and can not be said to exist. As has been explained in this paper, I take that stance one step further and deny the validity as numbers of even the rationals, allowing only the positive integers as numbers.

As to what comes next there are two possible paths: we may continue as we are, inventing more and more uncompleted operations and defining them as numbers, or, we may go back to the first concepts of math and try to build a better understanding of what we have created. In doing so I can only conclude we would discover and invent a much richer subject than we have imagined to date.

Section 9: First Steps Towards a Positive Integer Mathematics

We Actually Already Have Two Systems:

We know that when counting children in a kindergarten class the result must be a positive integer. If we have too few chairs for the class we don't insist that 1.238095238... children sit in each chair. Likewise if we have too many chairs we don't have 0.8387096774... of a child in each chair. As simple as this observation is, we don't have a mathematics that actually tells us what to do in either case. We have an informal discrete positive integer system which we use instinctively when necessary. We would not tell the custodian that we have a ratio of children to chairs of 1.238095238..., that would not convey the needed information. We tell him we need 5 more chairs. Our infinitely divisible system of real numbers leads us astray when we use it to calculate numbers of discrete objects. Interestingly, we know how to calculate the number of chairs needed: we subtract chairs from children. Say we have 26 children and only 21 chairs, we know we need 5 more chairs *by doing an illegal operation* of subtracting unlike objects: chairs from children. That is the same as 'adding apples to oranges'. If instead of using subtraction to solve for the number of chairs needed we used division we arrive at the real numbers indicated above. For example, we want the ratio of chairs to children to be exactly one, so we divide 21 chairs by 26 children and obtain 0.83…. We can see the result is less than one but we can't tell how many chairs we need.

An alternative approach is to realize that the needed ratio of chairs to children is 1 chair / 1 child. If we have 26 children we can multiply the number of children times the needed ratio to determine how many chairs are needed, then compare that to the number of chairs available: 26 children * 1 chair / 1 child = 26 chairs. Now we can subtract chairs from chairs: 26 chairs – 21 chairs = 5 needed chairs. But notice we are still not really subtracting like objects: 26 *needed chairs* – 21 *available chairs* = 5 needed chairs. To further show the difference between the two sets of objects, available chairs are real objects, whereas needed chairs are conceptual (in the mind) objects.

Notice that if we add the chairs to the children we obtain 47 'objects', where *any given object* is either a chair or a child. And if we find the difference between chairs and children we get 5 objects, where *all objects* must be either chairs or children. A subtle but significant distinction. It is unlikely we would ask the custodian to send us 5 more children, or a mix of objects – some chairs, some children. Notice that I said 'find the difference', not subtract. If we subtract chairs from children we may obtain a negative result which would indicate a surplus of chairs, whereas a positive result would indicate too few chairs. A new operation is needed here, 'dom', which we will explore shortly.

From the preceding paragraphs it's apparent we need to formalize our approach to discrete mathematics. That is, we must supplement our existing mathematics with operations designed to work correctly when dealing with finite numbers of discrete indivisible elements.

Counting and Adding:

Counting naturally uses the positive integers starting with one, there is nothing that needs to be modified to incorporate counting into a PIM (positive integer mathematics). Addition is also naturally closed with the positive integers and can in incorporated with only a single modification. *The sum of two numbers must not exceed the total of available objects.*

<u>Subtraction and Negative Numbers:</u>

Here we find the first required shifts in perspective as subtraction of positive integers may lead to negative numbers, which we do not allow in a PIM. Negative numbers also arise from physical scales such as temperature in degrees centigrade, and from a similar concept of direction of travel, for example, if up is positive then down must be negative.

The last two cases are the easiest to resolve. If a physical measurement 'goes negative' we simply re-adjust the starting point of the scale. Change the measurements from centigrade to Kelvin, for example. (The Kelvin scale currently starts with zero, indicating no (the non-existence of) relative motion of the material's constituent atoms. It is possible that there is a smallest discrete amount of motion, a sort of plank velocity, that is the lowest energy state possible.)

Consider that zero degrees centigrade does not mean that the temperature does not exist, after all, the mercury has still expanded a certain distance in its column in the thermometer. Zero degrees just means on a scale of ordered symbols consisting of negative and positive integers, and zero, that the mercury is currently at symbol '0'. Instead of integers we could have used words for the degree symbols, picking, perhaps, the word 'ice' for the position of the mercury column when water freezes.

As to the use of negative number for 'opposite direction' we just use different labels for different directions. For example, if 10 steps east is positive 10e, then rather than 10 steps west being -10e we let 10 steps west be positive 10w. We then adopt an algebra that allows for the addition and subtraction of 'e' and 'w' quantities. (See Section 7 Positive Axis Coordinate Systems.)

Negative numbers arising from subtraction of physical quantities indicates an error condition in the data or calculations. If you have 10 stones you cannot pretend to be able to subtract 12 stones. In this case the calculation is deemed invalid.

There may be some cases where during some steps of a calculation a value becomes negative but it is brought back to a positive value in subsequent steps. This may or may not reflect a valid application to the physical objects or events being measured. The case where a calculation is allowed to have uncompleted operations needs considerably more study; it allows, for example, the possibility of zero and division by zero as well as negative values.

If a calculation involves several variables each of which can take on any finite positive integer value then an expression like $x - y$ becomes a problem. Should a subtraction of positive integer variables appear in an equation then restrictions must be placed on the variables. The conditions of the equation and the restrictions must all be satisfied by a valid solution. For example:

$x - y = 3$

$y = x - 3$

if $x = 1$ then $y = -2 \rightarrow$ invalid result

restriction: $x > 3$, as well as $x - y = 3$, and, of course, PIM requirements of $x, y \geq 1$.

In real world applications I suspect that boundary conditions and other physical laws provide these additional restrictions so that the mathematical formulations generally work without our intervention. However, extrapolating an equation's uncompleted operation results to indicate actual real world values must be viewed as suspect. Simple things like negative directions can be translated into a positive opposite direction, but saying that the Lorentz transform equation *doesn't* limit the velocity of an object to less than the speed of light does not mean that imaginary velocities *are* actually obtainable.

Reduction:

The problems that arise from subtraction are due to the way the operation is defined. We could define an operation, reduction, whose definition is 'that quantity remaining when a given quantity is reduced by a quantity less than the given quantity'. Notice, by the way, that if we wish to define the outcome of reduction to be a quantity the definition must be strictly less than, not less than or equal to, the given quantity.

With this operation the need for negative numbers and zero never arises. Is this operation of any practical use? Well, yes and no. It certainly doesn't integrate well with the current use of formulas and equations. The equation a = b – c does not in any way require that c be less than b, and if it is not the result becomes negative (or zero) and we have developed ways to cope with that situation.

But if we are pragmatic about the application of mathematics to the real world the operation of reduction better fits the operation of a finite and discrete universe. Notice, we could include addition and reduction as discovered mathematics on the discovered positive integers while excluding all other operations and numbers as invented.

Dom:

The operation needed to solve the chairs and children problem mentioned above I call dom. This is taken directly from the concept of one set dominating another in number of elements. Rather than subtract chairs from children we 'dom' chairs *and* children. The result yields the difference and the type. Dom can be defined as the absolute value of the difference, *with the label of the largest set*. In practice, if we dom B chairs and A children the result is: |A-B| children iff A-B > 0, |A-B| chairs iff A-B < 0, and 'does not exist' if A-B = 0. About the only easy to access symbol in common fonts is the ◊ sign, which, for lack better, I'll use for this operation. Thus A◊B determines the difference and type between two sets of different types of elements. Notice that addition of apples and oranges can be said to be undefined or result in a different type, 'fruit', but the dom of apples and oranges results in either one or the other type (or no type). This is the subtle distinction mentioned above.

Zero:

Zero is not a number and is not used as such in any PIM system. In the standard base ten number system there are the usual ten digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. The '0' symbol in a standard-like PIM is not a number or a digit. It can only appear to the right of the most significant digit. Its meaning is simply and only, 'there is not a multiplying digit for this power of ten'.

This PIM system of nine digits, 1 – 9, and a place holder (not a digit or number), 0, is easier to work with than a bijective system using ten digits, 1 - #, and without a zero or place holder. Valid numbers in this PIM system include: 1, 12, 10, 107, 300030, and 3021. Invalid numbers include: any number with a radix point, 0, and any number with 0 as its leftmost digit.

A result of zero indicates 'does not exist', or in computer parlance, NaN (Not a Number).

The Role of Uncompleted Operations in Number Systems
Robert Duncan

Rational Numbers Recast:

Section 1, Rational Numbers, alludes to the problems associated with rational numbers as uncompleted operations. How must we deal with these artifacts of infinite and continuous logic in a finite and discrete world?

The first effort at rectification of a rational number should be counting or measuring the data with a different scale. This may be all that is necessary to reach the level of precision needed to engineer a solution. For example, instead of ½ quart we could use 1 pint.

If the problem deals with indivisible (discrete) objects such as marbles there are two methods of resolution. For example: 10 marbles to be given to 7 children. Each child gets 10/7 marbles. We can define new types of division. Standard division requires equal sized results, it is the inverse function of multiplication which always forms a result from addition of equal parts. However, we could define a division with remainder in which case each child gets 1 marble. We could define a division with unequal parts in which case 4 children get 1 marble and 3 children get 2 marbles. To our unconscious paradigm the first seems wasteful and the second seems unfair, but the children would not be happy with 1.42857… marbles each.

A third resolution is also painful to accept: it may not be possible to resolve some mathematical problems. We could tell the children that we can’t give them any marbles because the math doesn’t work.

Imaginary Numbers Recast:

We must divorce the imaginary number ‘*i*’ from the concept of ‘the square root of negative one’, and from the idea of ‘rotation’ in the plane. Apparently some processes in the universe require an algebra that can be simulated by standard algebraic operations on √-1, why this is the case begs for deeper analysis.

The rules of algebra for imaginary numbers are: imaginary numbers can be added to other imaginary numbers but not to real numbers (gives an uncompleted operation result); imaginary numbers can be multiplied with other imaginary numbers yielding a real number (negative if both imaginary numbers were positive); imaginary numbers can be multiplied with real numbers yielding imaginary numbers.

Imaginary numbers are easily defined on the complex plane, where multiplication by i causes a rotation of 90 degrees. But the rotation is only due to the juxtaposition of the real axis over the imaginary axis at right angles. Instead, picture a real axis going left and right, then above and parallel to the real axis an imaginary axis.

Let a, b, and c be numbers on the real axis and x, y and z be numbers on the imaginary axis. Then the sums a+b yields c which stays on the the real axis and x+y yields z which stays on the imaginary axis. But, summing a and x yields a+x: the two numbers remain on their respective axes with no single resolution. Similarly, a*b yields c on the real axis, but x*y yields c also on the real axis as well. However, a*x yields z on the imaginary axis.

Note that the type of numbers on the two axes are the same, real numbers, only the labels, real or imaginary are different. The point here is that the algebra forces jumps from one property or label to another, not that there exist different types of numbers.

A better picture of this situation is a positive real axis with a negative real axis above it, a positive imaginary axis above those, and negative imaginary axis above them all, with each axis numbered with positive integers but labeled differently, that is, quantifying different properties.

An even better picture is a separate positive integer axis with appropriate label for as many different entities or properties that are needed, along with an appropriate algebra.

Consider another example: the plane with the standard directions of north, south, east, and west. This is always pictured as a Cartesian plane with the north/south axis perpendicular to the east/west axis, but it need not be. Imagine the four axes stacked parallel to each other each running left to right, and numbered with positive integers starting (for now) with zero. Multiplication is not defined for these properties (the four directions) but addition is as follows: the sum of two values of the same direction is the normal sum in the same direction. The sum of two different directions that belong to the pair east/west or the pair north/south is the absolute value of the difference of the two with the direction of the larger (a 'dom' calculation). Two different directions that do not belong to either of the pairs east/west or north/south can not be added and remain as two directions. That is, there is no axis for northeast or north by northeast, or any such combinations.

Thus, a trip of 4 north plus 2 south results in 2 north, while a trip of 4 north plus 2 east is results in an uncompleted operation of 4 north plus 2 east. One hopes for the result to be a single number, that uncompleted operations will 'cancel out' of the final answer, and sometimes they do. A trip of 4 north plus 2 east plus 1 south plus 2 west results in a trip 3 north.

Irrational Numbers Recast:

The irrationals are best seen as ratios of incommensurate numbers, and can always be written as an irrational number over one. Pi is 3.1415…/1, √2 is √2/1, etc. where the one represents the unit that is incommensurate with the irrational. Pi, expressed as the ratio of a circle's circumference over its diameter 'd', is d*3.1415…/d; dividing numerator and denominator by d yields 3.1415…/1.

In a finite and discrete world there are no incommensurate numbers. The units of two measurements of the same dimension (feet, atoms, stones) are always reducible to the smallest finite unit; measurements of different dimensions (feet per second) cannot be reduced to a single label and must remain a rate (ratio). This suggests that distance around a circle (circumference) and distance across a circle (diameter) should have different units of measurement that do not cancel in a ratio.

There is a proviso in this argument: length as a distance through space, and time as a distance through time, may not have discrete units and may be infinitely divisible, but I doubt it.

Requirements of a Positive Integer Mathematics:

Positive integer math (PIM) is applicable to finite sets of discrete elements using operations that are complete in a finite number of steps. It is an underlying assumption that the universe and all things in it may be finite in size and number. A corollary to this assumption is that current mathematical constructs such as the irrational numbers, infinite and infinitesimal quantities, and the operations that make use of those quantities, among many other quantities and operations, are useful only in the special situations where very large numbers of objects (or operations) are approximated as a continuum of a few infinitely divisible conglomerates. Until quantum physics the current math worked well for almost every physical formulation (diophantine analysis addressed most of the exceptions).

A rigorous approach to mathematical equations involving physical quantities and properties would require the units of each quantity to be the smallest indivisible size applicable: instead of kilograms of gold we should work with atoms of gold; instead of meters we might use something like Plank lengths. In all cases the quantities would be represented by positive integers only. Negative, zero, rational, irrational, real, imaginary, and complex numbers would not be defined in such a rigorous approach, and all defined operations on the integers would be closed.

Fortunately, macro-world approximations do not require such rigor, and the fictions of non-integer numbers serve quite well in most predictive equations. However, such approximations may obscure the workings of the universe, and their success leads us to assume the universe is a reflection of our mathematics, rather than the more reasonable approach of trying to create a mathematics that reflects the properties and actions of the universe.

Stated assumptions:

- the elements of PIM are the positive integers (pure numbers) and dimensions
- pure numbers without labels are not extant – pure number mathematics is an interesting game that may not have any relevance to describing the physical universe
- any real physical thing has at least one dimension (unit, label) differentiating it from all other things (position in space-time, size, color, as examples)
- any real physical thing may be considered one of a number (positive integer) of real things with at least one similar dimension
- all real, physical things are discrete at a some scale (time and space may be as well)
- all real, physical things are finite in size and number (time and space may be as well)
- any physical thing is, in quantity, existent
    - if a thing exists it can be given a dimension, unit, or label
    - if a thing exists its quantity can be represented by a positive integer
    - if some presumed thing does not exist it has no quantity and can not be represented by a positive integer, it may, however, have a label (e. g. unicorns)
        - zero, commonly seen as a quantity, represents the state of not existing, *zero does not represent a quantity and is not a number*
- an existent physical thing (or set of things) is described, therefore, by two properties:
    - a positive integer quantity (a pure number) representing quantity, and
    - a unit or label (a dimension) differentiating that thing (those things) from all other things
- pure numbers represent quantities or counts and are not real physical things

Section #: Summary

There is obviously a considerable amount of work needed to turn these scattered ideas into a workable and useful mathematics.